Incompressible MHD simulations

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Simulation methods in astrophysics
Outline

- Where do we need incompressible MHD?
- Theory of HD&MHD
- Turbulence
- Numerical methods for HD
- Numerical methods for MHD
Closest available space plasma: 
The solar wind
- includes shocks
- and density fluctuation
The large scale evolution seems to be compressible...
...but in the frame of the solar wind, we find
- a turbulent spectrum
- Alfvén waves travelling

Sounds like incompressible turbulence!
Infamous picture
- this is usually used to suggest turbulence
- it really shows density fluctuations
- Density fluctuations? That’s compressible turbulence?
- Really?
Damping rates

Alfvén wave damping

Sound wave damping
So is this incompressible MHD?

- Wave damping suggests that for large $k$ only Alfvén waves survive
- Density fluctuations suggest there is compressible turbulence
- Two solutions:
  - Alfvén turbulence with enslaved density fluctuations
  - Not waves but shocks govern the ISM
Euler Equation

Force on a volume element of fluid

\[ F = - \oint p \, df \]  \quad (1)

Divergence theorem

\[ F = - \int \nabla p \, dV \]  \quad (2)

Newton’s law for one volume element

\[ m \frac{d\mathbf{u}}{dt} = F \]  \quad (3)

Using density instead of total mass

\[ \int \rho \frac{d\mathbf{u}}{dt} \, dV = - \int \nabla p \, dV \]  \quad (4)

\[ \rho \frac{d\mathbf{u}}{dt} = -\nabla p \]  \quad (5)

⇒ Lagrange picture
Euler Equation

Changing to material derivative

\[ d\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} dt + (d\mathbf{r} \nabla) \mathbf{u} \]  
\[ \Rightarrow \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} \]  

Euler’s equation

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla \rho \]  

Here the velocity is a function of space and time

\[ \mathbf{u} = \mathbf{u}(\mathbf{r}, t) \]  

This can now be applied to the momentum transport equation

\[ \frac{\partial}{\partial t} (\rho u_i) = \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} \]
Euler Equation

Partial derivative of $u$ is known from Euler

\[
\frac{\partial u_i}{\partial t} = -u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \tag{1}
\]

\[
\Rightarrow \frac{\partial}{\partial t} (\rho u_i) = -\rho u_k \frac{\partial u_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - u_i \frac{\partial \rho u_k}{\partial x_k} \tag{2}
\]

\[
= -\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k} (\rho u_i u_k) \tag{3}
\]

\[
= -\delta_{ik} \frac{\partial p}{\partial x_k} - \frac{\partial}{\partial x_k} (\rho u_i u_k) \tag{4}
\]

\[
\frac{\partial \rho u_i}{\partial t} = -\frac{\partial}{\partial x_k} \Pi_{ik} \tag{5}
\]

\[
\Pi_{ik} = -\rho \delta_{ik} - \rho u_i u_k \tag{6}
\]

$\Pi_{ik} =$ stress tensor
Continuity equation

\[
\frac{\partial}{\partial t} \int \rho \, dV = - \int \rho \mathbf{u} \, dV \tag{7}
\]

\[
\int \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) \, dV = 0 \tag{8}
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{9}
\]

For incompressible fluids (\(\rho=\text{const}\))

\[
\nabla \cdot \mathbf{u} = 0 \tag{10}
\]
Viscid fluid

\[ \Pi_{ik} = p\delta_{ik} + \rho u_i u_k - \sigma_{ik} \quad (11) \]

\( \sigma_{ik} = \) shear stress

For Newtonian fluids this depends on derivatives of the velocity

\[ \sigma_{ik} = a \frac{\partial u_i}{\partial x_k} + b \frac{\partial u_k}{\partial x_i} + c \frac{\partial u_l}{\partial x_l} \delta_{ik} \quad (12) \]

Coefficients have to fulfill

- no viscosity in uniform fluids (\( u = \text{const} \))
- no viscosity in uniform rotating flow

It follows \( a = b \)

\[ \sigma_{ik} = \eta \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial u_l}{\partial x_l} \right) + \zeta \delta_{ik} \frac{\partial u_l}{\partial x_l} \quad (13) \]
Viscid fluid

Stress tensor for incompressible fluids

\[
\frac{\partial u_i}{\partial x_i} = 0 \quad (11)
\]

\[
\Rightarrow \sigma_{ik} = \eta \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \quad (12)
\]

\[
\frac{\partial \sigma_{ik}}{\partial x_k} = \eta \frac{\partial^2 u_i}{\partial x_k^2} \quad (13)
\]

And we find the **Navier-Stokes-equation**:

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} \quad (14)
\]

\[
\nu = \frac{\eta}{\rho} = \text{Viscosity}
\]

Pressure

\[
\Delta p = -(\mathbf{u} \nabla) \mathbf{u} + \nu \Delta \mathbf{u} \quad (15)
\]
Using \( \nabla \cdot \mathbf{u} = 0 \) one can take the curl of the Navier-Stokes-equation, resulting in a PDE for \( \omega = \nabla \times \mathbf{u} \):

\[
\frac{\partial}{\partial t} (\nabla \times \mathbf{u}) - \nabla \times (\mathbf{u} \times (\nabla \times \mathbf{u})) = -\nu \nabla \times (\nabla \times (\nabla \times \mathbf{u})) \tag{16}
\]

\[
\Rightarrow \frac{\partial \omega}{\partial t} = (\omega \nabla) \mathbf{u} + \nu \Delta \omega \tag{17}
\]

\( \omega \) is quasi-scalar for 2d-velocities
We need the stream function \( \Psi \) to derive velocities

\[
\omega = \nabla \times \mathbf{u} \tag{18}
\]

\[
\mathbf{u} = \nabla \times \Psi \tag{19}
\]

\[
\Delta \Psi = -\omega \tag{20}
\]
The MHD equations

Compressible MHD

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]
\[
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p - \frac{1}{4\pi} \mathbf{B} \times \nabla \times \mathbf{B}
\]
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})
\]

Solutions

Different wave modes are solutions to the MHD equations

- Alfvén modes (incompressible, aligned to the magnetic field)
- Fast magnetosonic (compressible)
- Slow magnetosonic (compressible)
The MHD equations

Incompressible MHD

\[ \nabla \cdot \mathbf{v} = 0 \]
\[ \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p - \frac{1}{4\pi} \mathbf{B} \times \nabla \times \mathbf{B} \]
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \]

Solutions

Different wave modes are solutions to the MHD equations

- Alfvén modes (incompressible, aligned to the magnetic field)
The MHD equations

Elsasser variables

\[ w^- = v + b - v_A e_z \]
\[ w^+ = v - b + v_A e_z \]
\[ \Rightarrow v = \frac{w^+ + w^-}{2} \]
\[ \Rightarrow b = \frac{-w^+ + w^- - 2v_A e_z}{2} \]

Solutions

Different wave modes are solutions to the MHD equations

- Alfvén modes (incompressible, aligned to the magnetic field)
- \( w^\pm \) correspond to forward/backward moving waves
Short review of Kolmogorov

- Kolmogorov assumes energy transport which is scale invariant
- Time-scale depends on eddy-turnover times
- Detailed analysis of the units reveals 5/3-law

Simplified picture – but enough to illustrate what we are doing next
Kolmogorov assumes energy transport which is scale invariant
- Time-scale depends on eddy-turnover times
- Detailed analysis of the units reveals 5/3-law

Simplified picture – but enough to illustrate what we are doing next

**Dimensional Argument**

5/3-law can be derived by assuming scale independent energy transport and dimensional arguments. This gives a unique solution.
• General idea: Turbulence is governed by Alfvén waves
• Collision of Alfvén waves is mechanism of choice
• Eddy-turnover time is then replaced by Alfvén time scale
Energy $E = \int (V^2 + B^2) dx$ is conserved
Cascading to smaller energies
Dimensional arguments cannot be used to derive $\tau_\lambda = \lambda / \nu_\lambda$
Dimensionless factor $\nu_\lambda / \nu_A$ can enter
Kraichnan-Iroshnikov

Wave-packet interaction

\[ w^- = v + b - v_A e_z \]
\[ w^+ = v - b + v_A e_z \]

Solutions: \[ w^\pm = f(r \mp v_A t) \]

Figure: Packets interact, energy is conserved, shape is not
Packet interaction

Amplitudes $\delta w^+ \propto \delta w^- \propto \delta v_\lambda \propto \delta b_\lambda$
during one collision

$$\Delta \delta v_\lambda \propto (\delta v_\lambda^2 / \lambda)(\lambda / v_A)$$

Number of collision required to change wave packet

$$N \propto (\delta v_\lambda / \Delta \delta v_\lambda)^2 \propto (v_A / \delta v_\lambda)^2$$

$$\tau_{IK} \propto N \lambda / v_A \propto \lambda / \delta v_\lambda (v_A / \delta v_\lambda)$$

$$E_{IK} = \delta v_k^2 k^2 \propto k^{-3/2}$$

$\delta w^+ \propto \delta w^- \propto \delta v_\lambda \propto \delta b_\lambda$

$\Delta \delta v_\lambda \propto (\delta v_\lambda^2 / \lambda)(\lambda / v_A)$
• 3/2 spectrum not that often observe
• Let’s get a new theory
• We come back to Kraichnan-Iroshnikov: Alfvén waves govern the spectrum
• Turbulent energy is transported through multi-wave interaction
• Goldreich and Sridhar made up a number of articles dealing with this problem
• They distinguish between strong and weak turbulence
Weak Turbulence

What is weak turbulence?

Weak turbulence describes systems, where waves propagate if one neglects non-linearities. If one includes non-linearities wave amplitude will change slowly over many wave periods. The non-linearity is derived pertubatevily from the interaction of several waves.
Goldreich-Sridhar claim that Kraichnan-Iroshnikov can be seen as three-wave interaction of Alfvén waves.

They also believe, this is a forbidden process, since resonance condition reads:

\[ k_{z1} + k_{z2} = k_z \]

\[ |k_{z1}| + |k_{z2}| = |k_z| \]

4-wave coupling is now their choice!
Weak Turbulence

\[
\begin{align*}
k_1 + k_2 &= k_3 + k_4 \\
\omega_1 + \omega_2 &= \omega_3 + \omega_4 \\
\Rightarrow k_{1z} + k_{2z} &= k_{3z} + k_{4z} \\
k_{1z} - k_{2z} &= k_{3z} - k_{4z}
\end{align*}
\]

Parallel component unaltered $\Rightarrow$ 4-wave interaction just changes $k_\perp$ of quasi-particles
Goldreich-Sridhar

Weak Turbulence

\[ |\delta v_\lambda| \sim \left| \frac{d^2 v_\lambda}{dt^2} (k_z v_A)^{-2} \right| \]

\[ \frac{d^2 v_\lambda}{dt^2} \sim \frac{d}{dt} (k_\perp v_\lambda^2) \sim k_\perp v_\lambda \frac{dv_\lambda}{dt} \sim k_\perp^2 v_\lambda^3 \]

since \((v \cdot \nabla) \sim v_\lambda k_\perp\) for Alfvén waves

\[ \left| \frac{\delta v_\lambda}{v_\lambda} \right| \sim \left( \frac{k_\perp v_\perp}{k_z v_\lambda} \right)^2 \Rightarrow N \sim \left( \frac{k_\perp v_\perp}{k_z v_\lambda} \right)^4 \]

\[ \sum v_\lambda^2 = \int E(k_z, k_\perp) d^3 k \]

constant rate of cascade \[ E(k_z, k_\perp) \sim \epsilon(k_z) v_a k_\perp^{-10/3} \]

This spectrum now hold, when the velocity change is small
**Strong Turbulence**

Assume perturbation $v_\lambda$ on scales $\lambda_\parallel \sim k_z^{-1}$ and $\lambda_\perp \sim k_\perp^{-1}$

$$\zeta_\lambda \sim \frac{k_\perp v_\lambda}{k_z v_\lambda} \quad \text{Anisotropy parameter}$$

$$N \sim \zeta_\lambda^{-4} \sim k_\perp \sim k_z \sim L^{-1} \left( \frac{v_A}{v_L} \right)^4 (k_\perp L)^{-4/3}$$

Everything is fine as long as $\zeta_\lambda \ll 1$

GS assume *frequency renormalization* when this is not given
Strong Turbulence
When energy is injected isotropically $\zeta_L \sim 1$ we have a critical balance
\[ k_z v_A \sim k_\perp v_\lambda \]
\[ \Rightarrow \text{Alfvén timescale and cascading time scale match} \]
For the critical balance and scale-independent cascade we find
\[ k_z \sim k_\perp^{2/3} L^{-1/3} \]
\[ v_\perp \sim v_A (k_\perp L)^{-1/3} \]
\[ \Rightarrow E(k_\perp, k_z) \sim \frac{v_A^2}{k_\perp^{10/3} L^{1/3}} f \left( \frac{k_z L^{1/3}}{k_\perp^{2/3}} \right) \]
Goldreich-Sridhar

Strong Turbulence

Consequences

- A cutoff scale exists
- Eddies are elongated

Figure: From Cho, Lazarian, Vishniac (2003)
There is a number of turbulence models out there

- It is not yet clear which is correct
- All analytical turbulence models are incompressible
- The only thing we know for sure is that there is power law in the spectral energy distribution
Methods for the Navier-Stokes-equation

Projection  solve compressible equations, project on incompressible part  
Vorticity    use vector potential to always fulfill divergence-free condition  
Spectral     use Fourier space to do the same as above

Methods are presented for the Navier-Stokes-equation but also apply to MHD
Spectral methods will be presented for Elsässer variables in detail
Different possible schemes
Here a scheme proposed by [1].

**Step 1: Intermediate Solution**

We are looking for a solution $u^*$, which is not yet divergence free

$$\frac{u^* - u^n}{\Delta t} = -((u \nabla u)^{n+1/2} - \nabla p^{n-1/2} + \nu \Delta u^n)$$

(21)

This requires some standard scheme to solve the Navier-Stokes-equation (cf. Pekka’s talk)
Step 2: Dissipation

Use Crank-Nicholson for stability in the dissipation step

\[ \nu \Delta u^n \rightarrow \frac{1}{2} \nu \Delta (u^n + u^*) \] (21)

yields

\[ u^{**} = u^n - \Delta t ((u\nabla)u + \nabla p) \] (22)

\[ \left(1 - \frac{\nu \Delta t}{2} \Delta \right) u^* = \left(1 + \frac{\nu \Delta t}{2} \Delta \right) u^{**} \] (23)
Step 3: Projection step

To project $u^*$ on the divergence-free part, an auxiliary field $\phi$ is calculated, which fulfills

\begin{align}
\nabla \cdot (u^* - \Delta t \nabla \phi) &= 0 \quad (21) \\
\Delta \phi &= \frac{\nabla \cdot u^*}{\Delta t} \quad (22) \\
u^{n+1} &= u^* - \Delta t \nabla \phi^{n+1} \quad (23)
\end{align}

Here a Poisson equation has to be solved. This is the tricky part and is the most time consuming.
Step 4: Pressure gradient

Finally the pressure is updated. This can be calculated using the divergence of the Navier-Stokes-equation

\[ p^{n+1/2} = p^{n-1/2} + \phi^{n+1} - \frac{\nu \Delta t}{2} \Delta \phi^{n+1} \]  (21)
Total scheme

\[
\frac{u^* - u^n}{\Delta t} + \nabla p^{n+1/2} = -[(u \cdot \nabla)u]^{n+1/2} + \frac{\nu}{2} \nabla^2 (u^n + u^*) \tag{21}
\]

\[
\Delta t \nabla^2 \phi^{n+1} = \nabla \cdot u^* \tag{22}
\]

\[
u^n = u^* - \Delta t \nabla \phi^{n+1} \tag{23}
\]

\[
\nabla p^{n+1/2} = \nabla p^{n-1/2} + \nabla \phi^{n+1} - \frac{\nu \Delta t}{2} \nabla \nabla^2 \phi^{n+1} \tag{24}
\]
The natural formulation for divergence-free flows is the stream function $u = \nabla \times \psi$

**Step 1: Calculate velocity**

$$u^n = \nabla \times \psi^n$$ (25)
Step 2: Solve the Navier-Stokes-equation

$$\partial_t u = - (u \nabla) u + \nu \Delta u$$ (25)
Step 3: Calculate Vorticity

\[ \partial_t \omega = \nabla \times (\partial_t u) \quad (25) \]
\[ \omega^{n+1} = \omega^n + dt \cdot \Delta \omega \quad (26) \]
Step 4: Calculate Streamfunction

Solve Poisson equation

\[ \Delta \psi^{n+1} = -\omega^{n+1} \]  \hspace{1cm} (25)

Problem: This would result in a mesh-drift instability. Use staggered grid!
**Complete scheme**

\[
\begin{align*}
\mathbf{u} &= \nabla \times \Psi \quad (25) \\
(u_x)_{i,j+1/2} &= \frac{\Psi_{i,j+1} - \Psi_{i,j}}{dy} \quad (26) \\
(u_y)_{i+1/2,j} &= \frac{\Psi_{i+1,j} - \Psi_{i,j}}{dx} \quad (27) \\
u_{i+1/2,j+1/2} &= \frac{1}{2} \left( (u_x)_{i+1,j+1/2} + (u_x)_{i,j+1/2} \right) \\
&\quad \frac{1}{2} \left( (u_y)_{i+1/2,j+1} + (u_y)_{i+1/2,j} \right) \quad (28)
\end{align*}
\]
Spectral methods

Basic idea: Use Fourier transform to transform PDE into ODE
Problem: This is complicated for the nonlinear terms.

Spectral Navier-Stokes-equation

\[
\frac{\partial \tilde{u}_\alpha}{\partial t} = -i k_\gamma \left( \delta_{\alpha \beta} - \frac{k_\alpha k_\beta}{k^2} \right) (\tilde{u}_\beta \tilde{u}_\gamma) - \nu k^2 \tilde{u}_\alpha
\]

(29)

\[k_\alpha \tilde{u}_\alpha = 0\]

Quantities with a tilde are fouriertransformed.
What’s so special about the red and green term?
Spectral methods

The green term

Start with the Navier-Stokes-equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} + \nabla \rho = 0$$  \hspace{1cm} (29)

we can identify

$$i k_{\gamma} \delta_{\alpha \beta} \hat{u}_\beta u_\gamma = (\mathbf{u} \nabla) \mathbf{u}$$  \hspace{1cm} (30)

and taking the divergence of the Navier-Stokes-equation

$$\Delta \rho = \nabla \cdot (\mathbf{u} \nabla) \mathbf{u}$$  \hspace{1cm} (31)

so the second term is the gradient of the pressure

$$i k_{\gamma} \left( \frac{k_\alpha k_\beta}{k^2} \right) (\hat{u}_\beta u_\gamma) = -\nabla \rho$$  \hspace{1cm} (32)
Step 1: Fourier transform

Starting with $\tilde{u}_\alpha$ the velocity has to be transformed into $u_\alpha$

Then $u_\alpha u_\beta$ can be calculated (only 6 tensor elements are needed)

This is transformed back to Fourier space $\tilde{u}_\alpha u_\beta$
Step 2: Anti-Aliasing

In the high wavenumber regime of $u_\alpha$ we will find flawed data. Due to the fact, that not all data is correctly attributed for by the Fourier transform, aliased data has to be removed.

For $|k| > 1/2k_{\text{max}}$ set $\widehat{u_\alpha u_\beta}(k) = 0$
Step 3: Update velocity

\[ \tilde{u}_\alpha^* = \tilde{u}_\alpha^n - \Delta t (\hat{k}_\gamma \left( \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) (\overline{u_\beta u_\gamma}) - \nu k^2 \tilde{u}_\alpha ) \]  

(29)
Step 4: Projection

Project $u^*_\alpha$ on its divergence free part

$$u^{n+1} = u^* - \frac{k(k \cdot u^*)}{k^2}$$

(29)
Why to use Spectral Methods?

**Pro**
- Easy to implement
- Relies on fast FFT-algorithms
- Easy to parallelize

**Contra**
- high aliasing-loss
Fourier transforms are available in large numbers on the market

- **FFTW 2** ubiquitious library, allows parallel usage
- **FFTW 3** the improved version, *no parallel support*
- **Intel MKL** includes DFT, much faster than FFTW, simple parallelized version, limited to Intel-like systems
- **P3DFFT** UC San Diego’s Fortran based FFT, highly parallelizable, requires FFTW-3
- **Sandia FFT** Sandia Lab’s C based FFT, highly parallelizable, requires FFTW-2
Usually a parallel fluid simulation requires exchange of field borders between processors.
Spectral methods are slightly different:
- The calculation of $u_\alpha u_\beta$ and the update step are completely local. They have no derivatives and do not need neighboring points
- Every non-local interaction is done in the FFT
So the only thing we have to do, is using local coordinates and a parallel FFT
The equations are treated similarly to the Navier-Stokes-equation.
Two major changes:

- Two coupled equations
- Magnetic field introduced via $-v_A k_z$ term