

Chapter 2

Basics of electrodynamics

2.1 The Maxwell equations

The Maxwell equations are

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0 \quad (2.1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.2)$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (2.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (2.4)$$

where *all* charges and currents are described by ρ and \mathbf{J} . Alternatively, the Maxwell equations in the "material form" are

$$\nabla \cdot \mathbf{D} = \rho \quad (2.5)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.6)$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (2.7)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (2.8)$$

Here the source terms are external ("free") charges ρ and external ("free") currents \mathbf{J} (it is conventional to use the same symbols for them as in the "vacuum equations", although the interpretation is different). Charges and currents associated with polarization and magnetization are hidden in the fields \mathbf{D} ja \mathbf{H} , which are *defined* by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (2.9)$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad (2.10)$$

Here \mathbf{P} is the polarization (density of electric dipole moment vectors) and \mathbf{M} is the magnetization (density of magnetic dipole moment vectors).

We need to know the constitutive relations $\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B})$ and $\mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B})$, because the fields \mathbf{B} and \mathbf{E} are the ones which we finally want to determine. We will mostly deal with simple materials for which the following relations hold:

$$\mathbf{D} = \epsilon \mathbf{E} \quad (2.11)$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \quad (2.12)$$

We also assume that Ohm's law is valid in its simple form

$$\mathbf{J} = \sigma \mathbf{E} \quad (2.13)$$

Here permittivity, permeability and conductivity are scalars. We allow a spatial dependence with the limitation that the quantities are piecewise constant. It is worth remembering that there are much more complex media too, of which plasma or ferromagnetic materials are good examples.

The integral forms of the Maxwell equations are often useful:

$$\oint_S \mathbf{E} \cdot \mathbf{n} \, dS = \frac{1}{\epsilon_0} \int_V \rho \, dV \quad (2.14)$$

$$\oint_S \mathbf{B} \cdot \mathbf{n} \, dS = 0 \quad (2.15)$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, dS \quad (2.16)$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} \, dS + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot \mathbf{n} \, dS \quad (2.17)$$

Make yourself clear the meaning of C , S and V as well as the definitions of positive directions of \mathbf{n} and $d\mathbf{l}$.

To finish the basic review, we remind about the conservation of charge, from which the current continuity equation follows:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (2.18)$$

This is also obtained from the Maxwell equations. For a complete treatment we need the law of electromagnetic forces (Lorentz force):

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (2.19)$$

where q is the charge of a particle.

2.2 Boundary conditions

In the case that there are discontinuities in electromagnetic parameters, we need boundary conditions between two media. Let \mathbf{n} be the normal unit vector on a boundary pointing from material 1 to material 2. Then it follows from the integral form of the Maxwell equations that

1. The normal component of \mathbf{D} is discontinuous by the amount of a possible surface charge:

$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_s \quad (2.20)$$

where σ_s is the surface charge density (of charges which are not associated with polarization).

2. The normal component of \mathbf{B} is continuous:

$$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \quad (2.21)$$

3. The tangential component of \mathbf{E} is continuous:

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \quad (2.22)$$

4. The tangential component of \mathbf{H} is discontinuous by the amount of a possible surface current:

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K} \quad (2.23)$$

The surface current density \mathbf{K} is non-zero only on the surface of a perfect conductor.

It is possible to formulate other boundary conditions too. For example, $\partial\mathbf{B}/\partial t$ is continuous, because the tangential component of \mathbf{E} is continuous. It follows that with a harmonic time dependence the normal component of \mathbf{B} is continuous. So if the continuity of the tangential \mathbf{E} is already used the continuity of the normal \mathbf{B} does not provide any new information in the case of time-harmonic fields.

2.3 Potentials

2.3.1 Scalar and vector potentials

The homogeneous Maxwell equations 2.2-2.3 (or 2.6-2.7) imply that the electric and magnetic fields can be expressed in terms of the scalar and vector potentials φ and \mathbf{A} :

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (2.24)$$

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t} \quad (2.25)$$

The rest two equations (in the "vacuum" form) yield wave equations for the scalar and vector potentials:

$$\nabla^2\varphi + \frac{\partial(\nabla\cdot\mathbf{A})}{\partial t} = -\rho/\epsilon_0 \quad (2.26)$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) = -\mu_0\mathbf{J} \quad (2.27)$$

These equations are coupled, so they are not comfortable. However, a possible way to simplify them is to apply the Lorenz gauge condition¹:

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0 \quad (2.28)$$

Now the wave equations are decoupled:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = -\rho/\epsilon_0 \quad (2.29)$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu_0\mathbf{J} \quad (2.30)$$

The general solutions of the inhomogeneous wave equations are

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (2.31)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (2.32)$$

Solutions of the homogeneous equations (no source terms) may have to be added to them too. Here

$$t' = t - |\mathbf{r} - \mathbf{r}'|/c \quad (2.33)$$

is the retarded time taking into account the finite speed of the electromagnetic wave traveling from the source point \mathbf{r}' to the observation point \mathbf{r} .

We assumed above that the material is like vacuum. It is straightforward to show that in a uniform medium obeying Ohm's law the wave equations are

$$\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\right)\varphi = -\rho/\epsilon \quad (2.34)$$

$$\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\right)\mathbf{A} = -\mu\mathbf{J} \quad (2.35)$$

¹This is really the Lorenz gauge, not the Lorentz gauge. See Jackson, J.D. and L.B. Okun, Historical roots of gauge invariance, Rev. Mod. Phys., 73, 663-680, 2001.

where ϵ is permittivity, μ permeability and σ conductivity. Here \mathbf{J} denotes other than Ohmic currents. The Lorenz gauge takes now a generalised form:

$$\nabla \cdot \mathbf{A} + \mu\sigma\varphi + \mu\epsilon\frac{\partial\varphi}{\partial t} = 0 \quad (2.36)$$

An important special case is the harmonic time-dependence ($e^{-i\omega t}$), when the wave equations are

$$(\nabla^2 + \omega^2\mu\epsilon + i\omega\mu\sigma)\varphi(\mathbf{r}, \omega) = (\nabla^2 + k^2)\varphi(\mathbf{r}, \omega) = -\rho(\mathbf{r}, \omega)/\epsilon \quad (2.37)$$

$$(\nabla^2 + \omega^2\mu\epsilon + i\omega\mu\sigma)\mathbf{A}(\mathbf{r}, \omega) = (\nabla^2 + k^2)\mathbf{A}(\mathbf{r}, \omega) = -\mu\mathbf{J}(\mathbf{r}, \omega) \quad (2.38)$$

Then potentials can be calculated from the integrals

$$\varphi(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', \omega)}{|\mathbf{r} - \mathbf{r}'|} e^{ik|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (2.39)$$

$$\mathbf{A}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', \omega)}{|\mathbf{r} - \mathbf{r}'|} e^{ik|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (2.40)$$

The phase of the wave number

$$k = \sqrt{\omega^2\mu\epsilon + i\omega\mu\sigma} \quad (2.41)$$

is defined so that (draw a figure!)

$$\begin{aligned} 0 \leq \arg(k) \leq \pi/4 & \quad , \quad \omega > 0 \\ 3\pi/4 \leq \arg(k) \leq \pi & \quad , \quad \omega < 0 \end{aligned} \quad (2.42)$$

This convention guarantees the physical requirement that the integrals converge. It is a useful exercise to consider the phase convention in the case of a harmonic time-dependence $e^{+i\omega t}$. A lot of care is needed when dealing with these integrals, because they often lead to integration in the complex plane where phases must be thoroughly treated. When using integral tables, such restrictions must be carefully checked.

2.3.2 Hertz vectors

The scalar and vector potential φ and \mathbf{A} are not the only possibility to express the fields in terms of potentials. We will now present another representation, which is particularly convenient in some applications.

Assume first that the space is uniform and non-conducting, and that there are external charge and current distributions $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$. Define a vector $\mathbf{p} = \mathbf{p}(\mathbf{r}, t)$ so that

$$\begin{aligned} \rho &= -\nabla \cdot \mathbf{p} \\ \mathbf{J} &= \frac{\partial \mathbf{p}}{\partial t} \end{aligned} \quad (2.43)$$

These definitions are consistent with the equation of continuity, so the vector \mathbf{p} evidently exists. It follows immediately that the wave equations of the familiar scalar and vector potentials take the form

$$\begin{aligned}(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2})\varphi &= \frac{1}{\epsilon} \nabla \cdot \mathbf{p} \\ (\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2})\mathbf{A} &= -\mu \frac{\partial \mathbf{p}}{\partial t}\end{aligned}\quad (2.44)$$

Next define the electric Hertz vector $\mathbf{\Pi}$ so that

$$\begin{aligned}\varphi &= -\nabla \cdot \mathbf{\Pi} \\ \mathbf{A} &= \mu\epsilon \frac{\partial \mathbf{\Pi}}{\partial t}\end{aligned}\quad (2.45)$$

Note that this is consistent with the Lorenz gauge condition. It follows that

$$(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2})\mathbf{\Pi} = -\frac{1}{\epsilon} \mathbf{p} + \nabla \times \mathbf{C} \quad (2.46)$$

where the vector $\mathbf{C} = \mathbf{C}(\mathbf{r})$ depends only on spatial coordinates. Writing $\mathbf{C} = \nabla^2 \mathbf{G}(\mathbf{r})$ and defining $\mathbf{\Pi}' = \mathbf{\Pi} - \nabla \times \mathbf{G}$ we see that both $\mathbf{\Pi}$ and $\mathbf{\Pi}'$ satisfy the same equations. Consequently, it is possible to choose $\mathbf{C} = 0$. It follows that the fields can be expressed as

$$\begin{aligned}\mathbf{E} &= \nabla \times (\nabla \times \mathbf{\Pi}) - \frac{1}{\epsilon} \mathbf{p} \\ \mathbf{B} &= \mu\epsilon \nabla \times \frac{\partial \mathbf{\Pi}}{\partial t}\end{aligned}\quad (2.47)$$

An exercise is to derive the corresponding results in the case of a uniformly conducting medium:

$$\begin{aligned}\rho &= -\nabla \cdot \mathbf{p} \\ \mathbf{J} &= \frac{\partial \mathbf{p}}{\partial t} + \frac{\sigma}{\epsilon} \mathbf{p} \\ (\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2} - \mu\sigma \frac{\partial}{\partial t})\mathbf{\Pi} &= -\frac{1}{\epsilon} \mathbf{p} \\ \mathbf{E} &= \nabla \times (\nabla \times \mathbf{\Pi}) - \frac{1}{\epsilon} \mathbf{p} \\ \mathbf{B} &= \mu \nabla \times (\epsilon \frac{\partial \mathbf{\Pi}}{\partial t} + \sigma \mathbf{\Pi})\end{aligned}\quad (2.48)$$

If the charge density is zero then the current density is divergence-free: $\nabla \cdot \mathbf{J} = 0$. So it is possible to find a vector $\boldsymbol{\alpha}$ such that $\mathbf{J} = \nabla \times \boldsymbol{\alpha}$. The Maxwell equations imply that

$$\begin{aligned}\mathbf{E} &= -\nabla \times \mathbf{A}^* \\ \mathbf{B} &= \mu \boldsymbol{\alpha} - \nabla \varphi^* - \mu\epsilon \frac{\partial \mathbf{A}^*}{\partial t}\end{aligned}\quad (2.49)$$

where φ^* and \mathbf{A}^* are some scalar and vector functions (not related to the normal scalar and vector potentials). Next we define the magnetic Hertz vector $\mathbf{\Pi}^*$ so that

$$\begin{aligned}\varphi &= -\mu \nabla \cdot \mathbf{\Pi}^* \\ \mathbf{A} &= \mu \frac{\partial \mathbf{\Pi}^*}{\partial t}\end{aligned}\quad (2.50)$$

Then the Hertz vector satisfies the wave equation

$$(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}) \mathbf{\Pi}^* = -\alpha \quad (2.51)$$

The fields are

$$\begin{aligned}\mathbf{E} &= -\mu \nabla \times \frac{\partial \mathbf{\Pi}^*}{\partial t} \\ \mathbf{B} &= \mu \nabla \times (\nabla \times \mathbf{\Pi}^*)\end{aligned}\quad (2.52)$$

Note the similarity to the presentation with the electric Hertz vector with basically changing the roles of \mathbf{B} and \mathbf{E} .

2.3.3 Energy of the electromagnetic field

The conservation of the energy of a combined system of charged particles *and* electromagnetic fields is expressed by Poynting's theorem

$$\frac{\partial u_{em}}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} \quad (2.53)$$

where Poynting's vector is defined by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \quad (2.54)$$

The conventional interpretation is that the energy density of the electromagnetic field is

$$u_{em} = w_E + w_M = \frac{1}{2} \mathbf{D} \cdot \mathbf{E} + \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \quad (2.55)$$

The integral form of the conservation law is

$$\frac{d}{dt} (W_{mech} + W_{em}) = - \int_{\partial V} \mathbf{S} \cdot d\mathbf{A} \quad (2.56)$$

where W_{mech} is the mechanic energy of charged particles and W_{em} is the energy of the electromagnetic field. An exercise is to show that Poynting's theorem for time-harmonic ($e^{-i\omega t}$) fields is

$$\frac{1}{2} \int \mathbf{J}^* \cdot \mathbf{E} d^3\mathbf{r} + 2i\omega \int (w_E - w_M) d^3\mathbf{r} = - \oint \mathbf{S} \cdot d\mathbf{A} \quad (2.57)$$

where the asterisk denotes the complex conjugate and Poynting's vector and energy densities are now defined by

$$\begin{aligned}\mathbf{S} &= \frac{1}{2}\mathbf{E} \times \mathbf{H}^* \\ w_E &= \frac{1}{4}\mathbf{E} \cdot \mathbf{D}^* \\ w_M &= \frac{1}{4}\mathbf{B} \cdot \mathbf{H}^*\end{aligned}\tag{2.58}$$