Chapter 3

Radiating systems in free space

Electromagnetic waves are always generated by temporal changes of charge and current distributions. This chapter deals with the primary fields of such sources, i.e. there are no boundaries between different materials. Chapter 4 introduces scattering of primary waves from material bodies. Chapter 5 deals basically with scattering too, but at a quasi-static limit. A useful reference for Chapters 3-4 is *Jackson*. As mathematical tools, we need some experience with spherical harmonics and spherical Bessel and Neumann functions. Again, mathematical methods are common to several other branches of physics, for example quantum mechanics.

3.1 Multipole expansion of the vector potential

Assume that in an otherwise empty space there are charge and current distributions $\rho(\mathbf{r})e^{-i\omega t}$ and $\mathbf{J}(\mathbf{r})e^{-i\omega t}$. A general time dependence follows from the inverse Fourier transform. The vector potential in the Lorenz gauge is

$$\mathbf{A}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',\omega)}{|\mathbf{r}-\mathbf{r}'|}\, e^{ik|\mathbf{r}-\mathbf{r}'|}\, d^3\mathbf{r}' \tag{3.1}$$

where $k = \omega/c$. The magnetic field is $\mathbf{B} = \nabla \times \mathbf{A}$ and outside of the source region the electric field is $\mathbf{E} = ic^2\nabla \times \mathbf{B}/\omega$. So it is not necessary to determine the scalar potential. Note that it would be obtained readily from the Lorenz gauge condition $\nabla \cdot \mathbf{A} - i\omega\mu_0\epsilon_0\varphi = 0$.

Next we study sources whose length scales (d) are much smaller than the wavelength $\lambda = 2\pi/k = 2\pi c/\omega = c/f$. Then the space can be divided

into three different regions:

$$\begin{aligned} \text{near (static) zone:} \quad & d \ll r \ll \lambda \\ \text{intermediate (induction) zone:} \quad & d \ll r \sim \lambda \\ \text{far (radiation) zone:} \quad & d \ll \lambda \ll r \end{aligned}$$

For the near zone the exponential in the integral of the vector potential can be replaced by unity. Consequently, apart from the harmonic time dependence, the spatial behaviour of the vector potential is identical to the static case, and can be expanded into a series of spherical harmonics:

$$\mathbf{A}(\mathbf{r}) = \sum_{l,m} \mathbf{C}_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \tag{3.2}$$

In the far zone $kr \gg 1$, so the exponential term in the integral oscillates rapidly. The distance $|\mathbf{r} - \mathbf{r}'|$ can be replaced by $|\mathbf{r} - \mathbf{r}'| \approx r - \mathbf{n} \cdot \mathbf{r}'$, where $\mathbf{n} = \mathbf{r}/r$. The vector potential is now

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 e^{ikr}}{4\pi r} \int \mathbf{J}(\mathbf{r}', \omega) e^{-ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}' \tag{3.3}$$

If the source dimensions are small compared to the wavelength then it is reasonable to expand this expression into a power series with respect to k:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 e^{ikr}}{4\pi r} \sum_{m=0}^{\infty} \frac{(-ik)^m}{m!} \int \mathbf{J}(\mathbf{r}')(\mathbf{n} \cdot \mathbf{r}')^m d^3\mathbf{r}' \tag{3.4}$$

So the leading behaviour is $\mathbf{A} \sim e^{ikr}/r$. This is a spherical wave with an angular dependent coefficient. It is an exercise to show that the fields are transverse to $\mathbf{r}$ and fall off as $1/r$, corresponding to radiation fields.

The intermediate region is difficult, since the approximations made above are not possible. Leaving mathematical details as an exercise, we give the result

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} = ik \sum_{l=0}^{\infty} j_l(kr_<) h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \tag{3.5}$$

where j_l and $h_l^{(1)}$ are spherical Bessel and Hankel functions, respectively, and $r_< = min(r, r'), r_> = max(r, r')$. Then the expansion of the vector potential, valid for all r outside the source, is

$$\mathbf{A}(\mathbf{r}) = i\mu_0 k \sum_{l,m} h_l^{(1)}(kr) Y_{lm}(\theta, \phi) \int \mathbf{J}(\mathbf{r}') j_l(kr') Y_{lm}^*(\theta', \phi') d^3\mathbf{r}' \tag{3.6}$$

The mathematical usefulness of this expansion is the explicit separation of the observation and the source points in spherical coordinates.

3.1.1 Electric dipole field

Consider the 0th term of Eq. 3.4:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 e^{ikr}}{4\pi r} \int \mathbf{J}(\mathbf{r}') d^3\mathbf{r}' \tag{3.7}$$

Remembering the continuity equation $\nabla \cdot \mathbf{J} = i\omega\rho$ and taking into account that $\mathbf{J}$ is non-zero only in a finite volume, we obtain by integration by parts

$$\mathbf{A}(\mathbf{r}) = -\frac{i\omega\mu_0 e^{ikr}}{4\pi r} \int \mathbf{r}\rho(\mathbf{r}) d^3\mathbf{r} \tag{3.8}$$

Using the definition of the electric dipole moment from electrostatics ($\mathbf{p} = \int \mathbf{r}\rho(\mathbf{r}) d^3\mathbf{r}$), this can be written as

$$\mathbf{A}(\mathbf{r}) = -\frac{i\omega\mu_0 e^{ikr}}{4\pi r}\mathbf{p} \tag{3.9}$$

An inspection of Eq. 3.6 shows that this is the exact form of the first term everywhere outside the source, not only in the far region.

An exercise is to calculate the fields

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= k^2(\mathbf{n}\times\mathbf{p})\,\frac{\mu_0 c e^{ikr}}{4\pi r}(1-\frac{1}{ikr}) \\ \mathbf{E}(\mathbf{r}) &= k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\,\frac{e^{ikr}}{4\pi\epsilon_0 r} + (3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p})(\frac{1}{r^3}-\frac{ik}{r^2})\frac{e^{ikr}}{4\pi\epsilon_0} \end{aligned} \tag{3.10}$$

where $\mathbf{n} = \mathbf{r}/r$. At the static limit, the magnetic field vanishes and the electric field takes its familiar static dipole form. In the far zone, the radiation fields are

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= k^2(\mathbf{n}\times\mathbf{p})\,\frac{\mu_0 c e^{ikr}}{4\pi r} \\ \mathbf{E}(\mathbf{r}) &= c\mathbf{B}\times\mathbf{n} \end{aligned} \tag{3.11}$$

The fields are transverse to the radius vector from the source to the observation point, and they are also transverse to each other, and $|\mathbf{E}| = c|\mathbf{B}|$.

The radiated power is calculated using the Poynting vector. The meaningful quantity is the time-average, which for the harmonic time-dependence leads to the power per solid angle

$$\frac{dP_{rad}}{d\Omega} = \frac{1}{2\mu_0} Re(r^2\mathbf{n}\cdot\mathbf{E}\times\mathbf{B}^*) = \frac{\mu_0 c^3 k^4}{32\pi^2}|\mathbf{n}\times\mathbf{p}|^2 \tag{3.12}$$

The total power is

$$P_{rad} = \frac{\mu_0 c^3 k^4}{12\pi}|\mathbf{p}|^2 \tag{3.13}$$

3.1.2 Magnetic dipole field

The next term of the multipole expansion is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 e^{ikr}}{4\pi r}(\frac{1}{r} - ik) \int \mathbf{J}(\mathbf{r}')\ \mathbf{n}\cdot\mathbf{r}'\ d^3\mathbf{r}' \tag{3.14}$$

which is valid everywhere outside the source region. It is useful to separate the integrand into symmetric and antisymmetric parts with respect to $\mathbf{r}'$ and $\mathbf{J}$:

$$(\mathbf{n}\cdot\mathbf{r}')\mathbf{J} = \frac{1}{2}[(\mathbf{n}\cdot\mathbf{r}')\mathbf{J} + (\mathbf{n}\cdot\mathbf{J})\mathbf{r}'] + \frac{1}{2}(\mathbf{r}'\times\mathbf{J})\times\mathbf{n} \tag{3.15}$$

An exercise is to show that

$$\begin{aligned}\mathbf{A}(\mathbf{r}) = \frac{i\mu_0 k e^{ikr}}{4\pi r}(1 - \frac{1}{ikr})\ \mathbf{n}\times\mathbf{m} - \\ \frac{\mu_0 c k^2 e^{ikr}}{8\pi r}(1 - \frac{1}{ikr}) \int \mathbf{r}'(\mathbf{n}\cdot\mathbf{r}')\rho(\mathbf{r}')d^3\mathbf{r}'\end{aligned} \tag{3.16}$$

where $\mathbf{m}$ is the magnetic dipole moment of the current system:

$$\mathbf{m} = \frac{1}{2}\int \mathbf{r}\times\mathbf{J}(\mathbf{r})\ d^3\mathbf{r} \tag{3.17}$$

The first term of the vector potential has the same form as the magnetic field of the electric dipole field in the previous subsection. So the fields are obtained from the fields of the previous subsection leading to

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -k^2(\mathbf{n}\times\mathbf{m})\ \frac{\mu_0 c e^{ikr}}{4\pi r}(1 - \frac{1}{ikr}) \\ \mathbf{B}(\mathbf{r}) &= k^2(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\ \frac{e^{ikr}}{4\pi\epsilon_0 c r} + \\ &\quad (3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m})(\frac{1}{r^3} - \frac{ik}{r^2})\frac{e^{ikr}}{4\pi\epsilon_0 c r}\end{aligned} \tag{3.18}$$

The second term of the vector potential is more complicated. Since the integral involves second moments of the charge density, this term corresponds to an electric quadrupole. We will not study it further, and we also neglect all higher multipoles for which the present technique is tedious. A generally more powerful method deals with vector multipole fields (Sect. 3.3).

3.2 Examples of radiating systems

The previous section was somewhat abstract in considering the multipole expansion of the vector potential and related fields. Now we present some simple concrete radiating systems.

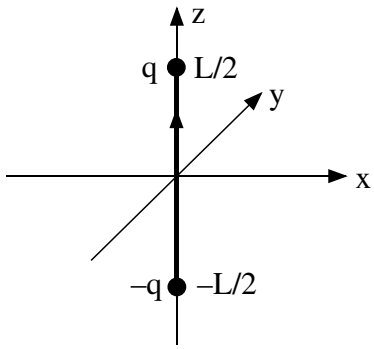

Figure 3.1: Simple dipole antenna.

3.2.1 Radiating dipole antenna

Consider an electric dipole consisting of two small spheres in the z axis at points $z = \pm L/2$ (Fig. 3.1). They are connected by a wire whose capacitance is negligible. If the charge of the upper sphere is $q(t)$ then the charge of the lower one is $-q(t)$. Conservation of charge yields the current density

$$\mathbf{J}(\mathbf{r}, t) = I(t)\delta(x)\delta(y)\theta(L/2 - z)\theta(z + L/2)\,\mathbf{e}_z \tag{3.19}$$

where $I = +dq/dt$. The vector potential has only the z component

$$A_z(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int_{-L/2}^{L/2} \frac{I(t - |\mathbf{r} - z'\mathbf{e}_z|/c)}{|\mathbf{r} - z'\mathbf{e}_z|}\,dz' \tag{3.20}$$

Before continuing, the reader should think qualitatively, which components the field has. Although this is dynamic system, impressions from electro- and magnetostatics are quite useful in this case.

At a large distances ($r \gg L$) we can approximate

$$|\mathbf{r} - z'\mathbf{e}_z| = (r^2 - 2z'\mathbf{e}_z \cdot \mathbf{r} + z'^2)^{1/2} \approx r - z'\cos\theta \tag{3.21}$$

where θ is the angle between the radius vector $\mathbf{r}$ of the observation point and the z axis. In the denominator of the vector potential, $z'\cos\theta$ can be ignored at large distances. In the retardation term it is negligible if $z'\cos\theta/c$ is small compared to the time scale of the current, for example, to the period T of a harmonically varying current. Since $z'\cos\theta \leq L/2$, we can omit $z'\cos\theta/c$ only if

$$L/2 \ll cT = \lambda \tag{3.22}$$

If this is the case then at large distances

$$A_z(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{L}{r} I(t - r/c) \tag{3.23}$$

The simplest way to determine the scalar potential is to apply the Lorenz gauge

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0 \tag{3.24}$$

yielding

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= -\frac{L}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left[\frac{1}{r}I(t-r/c)\right] \\ &= \frac{L}{4\pi\epsilon_0}\left[\frac{z}{r^3}I(t-r/c) + \frac{z}{r^2 c}I'(t-r/c)\right] \end{aligned} \tag{3.25}$$

where I' refers to differentiation of I with respect to $t - r/c$. On the other hand $I = +q'$, so

$$\varphi(\mathbf{r}, t) = \frac{L}{4\pi\epsilon_0}\frac{z}{r^2}\left[\frac{q(t-r/c)}{r} + \frac{I(t-r/c)}{c}\right] \tag{3.26}$$

Consider a harmonically oscillating dipole

$$\begin{aligned} q(t-r/c) &= q_0 \cos\omega(t-r/c) \\ I(t-r/c) &= I_0 \sin\omega(t-r/c) \;=\; -\omega q_0 \sin\omega(t-r/c) \end{aligned} \tag{3.27}$$

In spherical coordinates

$$\begin{aligned} A_r &= \frac{\mu_0}{4\pi}\frac{I_0 L}{r}\cos\theta \sin\omega(t-r/c) \\ A_\theta &= -\frac{\mu_0}{4\pi}\frac{I_0 L}{r}\sin\theta \sin\omega(t-r/c) \\ A_\phi &= 0 \end{aligned} \tag{3.28}$$

The magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ has now only a ϕ component

$$B_\phi = \frac{\mu_0}{4\pi}\frac{I_0 L}{r}\sin\theta\left[\frac{\omega}{c}\cos\omega(t-r/c) + \frac{1}{r}\sin\omega(t-r/c)\right] \tag{3.29}$$

The components of the electric field $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\varphi$ are

$$\begin{aligned} E_r &= \frac{2I_0 L\cos\theta}{4\pi\epsilon_0}\left[\frac{\sin\omega(t-r/c)}{r^2 c} - \frac{\cos\omega(t-r/c)}{\omega r^3}\right] \\ E_\theta &= -\frac{I_0 L\sin\theta}{4\pi\epsilon_0}\left[\left(\frac{1}{\omega r^3} - \frac{\omega}{rc^2}\right)\cos\omega(t-r/c) - \frac{1}{r^2 c}\sin\omega(t-r/c)\right] \\ E_\phi &= 0 \end{aligned} \tag{3.30}$$

It is easy to show that at large distances $\mathbf{E} = c\mathbf{B} \times \mathbf{e}_r$.

The radial component of the Poynting vector expresses the power radiated per a unit solid angle:

$$\frac{dP}{d\Omega} = R^2 E_\theta B_\phi/\mu_0 = \frac{1}{2\epsilon_0 c}\left(\frac{I_0 L\omega}{4\pi c}\right)^2 \sin^2\theta\cos^2\omega(t-R/c) \tag{3.31}$$

Integration over a spherical surface yields the total radiated power:

$$P = \oint \mathbf{S} \cdot \mathbf{n}\, da = \frac{1}{\mu_0} R^2 \int_0^\pi E_\theta B_\phi 2\pi \sin\theta\, d\theta \tag{3.32}$$

When $R \to \infty$, only the radiation fields proportional to $1/r$ contribute, so

$$P_{rad} = \oint \mathbf{S} \cdot \mathbf{n}\, da = \frac{(I_0 L)^2}{6\pi\epsilon_0} \frac{\omega^2}{c^3} \cos^2 \omega(t - R/c) \tag{3.33}$$

This is the instantaneous radiated power. Integration over the period $T = 2\pi/\omega$ provides the average power, which is generally a more relevant quantity:

$$\langle P_{ave} \rangle = \frac{L^2\omega^2}{6\pi\epsilon_0 c^3} \frac{I_0^2}{2} = \frac{2\pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{L}{\lambda}\right)^2 \frac{I_0^2}{2} \tag{3.34}$$

This is analogous to the average power $RI_0^2/2$ dissipated in an AC circuit whose resistance is R and current $I_0 \cos\omega t$. So it is meaningful to define the radiation resistance

$$R_r = \frac{2\pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{L}{\lambda}\right)^2 \approx 789 \left(\frac{L}{\lambda}\right)^2 \ \Omega \tag{3.35}$$

A magnetic dipole is analysed in the same way. It can be modelled as a circular loop carrying a time-harmonic current $I_0 \cos\omega t$. The dipole vector is perpendicular to the plane of the loop. Because the current has only the ϕ component, the only non-zero component of the vector potential is

$$A_\phi(\mathbf{r}, t) = \frac{\mu_0 I_0 a}{4\pi} \int_0^{2\pi} \frac{\cos\omega(t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} \cos\phi\, d\phi \tag{3.36}$$

where a is the radius of the loop. The dipole approximation presumes that $r \gg a$ and $\omega a \ll c$. The rest of the calculation is left as an exercise. A difference to an electric dipole is that the electric field is now tangential to any spherical surface centered at the current loop.

3.2.2 Half wavelength antenna

The result obtained in the previous section does not yield the correct radiation power of a true radio antenna, because an antenna is usually not short compared to the wavelength, and the current is typically fed into the centre, not to the ends.

Consider an antenna whose length is exactly equal to a half wavelength. This is a realistic example, since for example, the wavelength of a 100 MHz wave is 3 m. The antenna can be formally constructed of infinitesimal dipoles

considered in the previous section. Assume that the antenna is in the z-axis in $(-\lambda/4, +\lambda/4)$ and that its current is

$$I(z', t) = I_0 \sin \omega t \cos\left(\frac{2\pi z'}{\lambda}\right) \tag{3.37}$$

This is zero at both ends. The element at z' produces a radiation electric field whose θ-component is

$$dE_\theta = I_0 \frac{\sin\theta}{4\pi\epsilon_0 R c^2} \omega \cos\omega(t - R/c) \cos\left(\frac{2\pi z'}{\lambda}\right) dz' \tag{3.38}$$

Here R is the distance from dz' to the observation point, and terms of the order of $1/R^2$ are ignored. The ϕ component of the magnetic field is

$$dB_\phi = \frac{\mu_0}{4\pi} \frac{I_0 \omega}{Rc} \sin\theta \cos\omega(t - R/c) \cos\left(\frac{2\pi z'}{\lambda}\right) dz' \tag{3.39}$$

To get the total fields E_θ and B_ϕ, we must calculate the integral

$$K = \int_{-\pi/2}^{\pi/2} \frac{1}{R} \cos\omega(t - R/c) \cos u \, du \tag{3.40}$$

where $u = 2\pi z'/\lambda$. Again, $R = r - z' \cos\theta$ and at large r we can replace $1/R$ by $1/r$. The cosine term requires more care:

$$K \approx \frac{1}{r} \int_{-\pi/2}^{\pi/2} \cos[\omega(t - r/c) + u\cos\theta] \cos u \, du \tag{3.41}$$

This is equal to

$$K = \frac{1}{r} Re\,(e^{i\omega(t-r/c)} \int_{-\pi/2}^{\pi/2} e^{iu\cos\theta} \cos u \, du)$$

yielding

$$K = \frac{2}{r} \cos\omega(t - r/c) \frac{\cos[(\pi/2)\cos\theta]}{\sin^2\theta} \tag{3.42}$$

Substituting this gives the fields

$$E_\theta = \frac{I_0}{2\pi\epsilon_0 r c} \cos\omega(t - r/c) \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta} \tag{3.43}$$

$$B_\phi = \frac{\mu_0 I_0}{2\pi r} \cos\omega(t - r/c) \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta} \tag{3.44}$$

An exercise is to calculate the time-averaged radiated power:

$$\langle P \rangle = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} I_0^2 \int_0^\pi \frac{\cos^2[(\pi/2)\cos\theta]}{\sin^2\theta} \sin\theta \, d\theta \tag{3.45}$$

The integral is approximately 1,219, so the radiating power of a half-wavelength antenna is

$$\langle P \rangle \approx 73\,\Omega\, \frac{I_0^2}{2} \tag{3.46}$$

3.2.3 Centre-fed linear antenna

The third example is a thin linear antenna of length d excited by a coaxial cable across a small gap at its midpoint. The current density is

$$\mathbf{J}(\mathbf{r}) = I\sin(kd/2 - k|z|)\theta(|z| - d/2)\delta(x)\delta(y)\mathbf{e}_z \tag{3.47}$$

and the time-dependence is again harmonic. The vector potential in the far zone ($kr \gg 1$) is according to Eq. 3.3

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I e^{ikr}}{4\pi r}\int_{-d/2}^{d/2} \sin(kd/2 - k|z|)e^{-ikz\cos\theta}\, dz\ \mathbf{e}_z \tag{3.48}$$

resulting in

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I e^{ikr}}{2\pi kr}\ \frac{\cos(\frac{kd}{2}\cos\theta) - \cos(\frac{kd}{2})}{\sin^2\theta}\ \mathbf{e}_z \tag{3.49}$$

An exercise is to calculate the time-averaged radiation power per unit solid angle:

$$\frac{dP}{d\Omega} = \frac{I^2}{8\pi^2\epsilon_0 c}\ |\frac{\cos(\frac{kd}{2}\cos\theta) - \cos(\frac{kd}{2})}{\sin\theta}|^2 \tag{3.50}$$

3.2.4 Radiation due to a moving charged particle

The electromagnetic field due to a moving charged particle is familiar from the course of electrodynamics. We give a short overview of the results here.

The charge and current densities of a charged particle are

$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{r}_q(t)) \tag{3.51}$$

$$\mathbf{J}(\mathbf{r}, t) = q\dot{\mathbf{r}}_q(t)\delta(\mathbf{r} - \mathbf{r}_q(t)) \tag{3.52}$$

A straightforward way is to solve the inhomogeneous wave equations of the vector and scalar potentials in the Lorenz gauge using the method of Green's functions. The potentials are

$$\varphi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(1 - \mathbf{n}\cdot\boldsymbol{\beta})R}\right]_{ret} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R - \mathbf{R}\cdot\boldsymbol{\beta}}\right]_{ret} \tag{3.53}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\boldsymbol{\beta}}{(1 - \mathbf{n}\cdot\boldsymbol{\beta})R}\right]_{ret} = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\boldsymbol{\beta}}{R - \mathbf{R}\cdot\boldsymbol{\beta}}\right]_{ret} \tag{3.54}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_q$, $\mathbf{n} = \mathbf{R}/R$ and $\boldsymbol{\beta} = \mathbf{v}/c$. The subscript ret refers to the evaluation of the expressions at the retarded time t' defined by

$$t' + |\mathbf{r} - \mathbf{r}_q(t')|/c = t \tag{3.55}$$

The observer measures the fields at $\mathbf{r}$ at time t. A tedious differentiation yields the electric field

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{(1-\beta^2)(\mathbf{R}-R\boldsymbol{\beta}) + \mathbf{R}\times((\mathbf{R}-R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})/c}{(R-\boldsymbol{\beta}\cdot\mathbf{R})^3}\right]_{ret} \tag{3.56}$$

and the magnetic field

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\left[\frac{\mathbf{R}}{R}\right]_{ret}\times\mathbf{E}(\mathbf{r},t) \tag{3.57}$$

The electric field is basically a sum of a Coulomb field and a field due to acceleration of the particle. The former is of no further interest here, since its Poynting flux vanishes at infinity. The latter term is the radiation field vanishing as $1/R$. So the corresponding Poynting flux remains finite at infinity. In other words, radiation carries the field energy (and momentum and angular momentum) away from the particle.

The radiation fields of a slowly moving particle ($\beta \ll 1$) are

$$\mathbf{E}_{rad}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\,\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{v}})/R \tag{3.58}$$

$$\mathbf{B}_{rad}(\mathbf{r},t) = \frac{1}{c}\frac{\mathbf{R}}{R}\times\mathbf{E} = \frac{q}{4\pi\epsilon_0 c^3}\,\dot{\mathbf{v}}\times\mathbf{n}/R \tag{3.59}$$

Poynting's vector is

$$\mathbf{S} = \frac{1}{\mu_0}\,\mathbf{E}_{rad}\times\mathbf{B}_{rad} = \frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{|\mathbf{R}\times\dot{\mathbf{v}}|^2}{R^5}\mathbf{R} \tag{3.60}$$

This behaves as $1/R^2$, so the Poynting flux does not vanish. The radiation power per unit solid angle $d\Omega$ is

$$\frac{dP}{d\Omega} = \frac{q^2\dot{\mathbf{v}}^2}{16\pi^2\epsilon_0 c^3}\sin^2\theta \tag{3.61}$$

where θ is the angle between $\dot{\mathbf{v}}$ and $\mathbf{n}$. Integration yields the total power (Larmor's formula)

$$P = \frac{q^2\dot{\mathbf{v}}^2}{6\pi\epsilon_0 c^3} \tag{3.62}$$

For particles moving at high velocities, the difference between t and t' is significant. The radiated energy during the interval $t_1 = t_1' + R(t_1')/c ... t_2 = t_2' + R(t_2')/c$ is

$$W = \int_{t_1}^{t_2}[\mathbf{S}\cdot\mathbf{n}]_{ret}\,dt = \int_{t_1'}^{t_2'}\mathbf{S}\cdot\mathbf{n}\frac{dt}{dt'}dt' \tag{3.63}$$

It is meaningful to define the power radiated per unit area in terms of the charge's own time: $\mathbf{S}\cdot\mathbf{n}\, dt/dt' = \mathbf{S}\cdot\mathbf{n}\,(1-\mathbf{n}\cdot\boldsymbol{\beta})$. The power radiated per unit solid angle is then obtained in a straightforward manner using Poynting's vector resulting in

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c}\frac{\left|\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})\right|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5} \tag{3.64}$$

To interpret this, we may imagine that the particle is accelerated only for a short interval during which the velocity and acceleration vectors remain nearly constant. If we are far away so that $\mathbf{n}$ and R do not practically change during the acceleration interval then this formula gives the angular distribution of the radiated power.

If $\beta \to 1$ the effect of the denominator of $dP/d\Omega$ increases and the radiated energy flux concentrates more parallel to the velocity. The total radiated power is

$$P = \frac{q^2}{6\pi\epsilon_0 c}\gamma^6(\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2) \tag{3.65}$$

This result can be obtained in two ways, by a direct integration, or by using the relativistic formulation. However, we ignore the detailed calculation here.

3.2.5 Radiation due to a system of moving charged particles

Consider a set of slowly moving charges $v \ll c$ which are assumed to be far away from the observation point. More explicitly, all charges are within a volume V_1 for the time when the wave reaches the observer. Further, the scales of V_1 are assumed small compared to the wavelength and to the distance to the observer.

Let the origin be inside V_1, denote coordinates of the charges by $\mathbf{r}'$, of the observation point by $\mathbf{r}$, and define $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Now

$$R = |\mathbf{r}-\mathbf{r}'| \approx r - \frac{\mathbf{r}\cdot\mathbf{r}'}{r} \tag{3.66}$$

and the retarded scalar potential is

$$\begin{aligned}\varphi(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int_{V_1}\frac{\rho(\mathbf{r}',t-R/c)}{R}\,dV' \\ &\approx \frac{1}{4\pi\epsilon_0}\int_{V_1}\frac{\rho(\mathbf{r}',t-r/c+\mathbf{r}\cdot\mathbf{r}'/cr)}{r-(\mathbf{r}\cdot\mathbf{r}')/r}\,dV'\end{aligned} \tag{3.67}$$

Use of the binomial series

$$(r-\mathbf{r}\cdot\mathbf{r}'/r)^{-1} = r^{-1} + r^{-2}(\mathbf{r}\cdot\mathbf{r}'/r) + \ldots \tag{3.68}$$

and the Taylor expansion

$$\rho\left(\mathbf{r}', t - \frac{r}{c} + \frac{\mathbf{r}\cdot\mathbf{r}'}{cr}\right) = \rho\left(\mathbf{r}', t - \frac{r}{c}\right) + \frac{\mathbf{r}\cdot\mathbf{r}'}{cr}\frac{\partial\rho}{\partial t}\bigg|_{\mathbf{r}', t-r/c} + \dots \tag{3.69}$$

gives

$$\begin{aligned}\varphi(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0 r}\int_{V_1}\rho\left(\mathbf{r}', t - \frac{r}{c}\right)dV' + \frac{1}{4\pi\epsilon_0 r^3}\mathbf{r}\cdot\int_{V_1}\mathbf{r}'\rho\left(\mathbf{r}', t - \frac{r}{c}\right)dV' \\ &\quad + \frac{1}{4\pi\epsilon_0 r^2 c}\mathbf{r}\cdot\frac{d}{dt}\int_{V_1}\mathbf{r}'\rho\left(\mathbf{r}', t - \frac{r}{c}\right)dV' + \dots\end{aligned}$$

This is the familiar multipole expansion:

$$\varphi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\mathbf{r}\cdot\mathbf{p}(t-r/c)}{r^3} + \frac{\mathbf{r}\cdot\dot{\mathbf{p}}(t-r/c)}{cr^2}\right] \tag{3.70}$$

The retarded vector potential is

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int_{V_1}\frac{\mathbf{J}(\mathbf{r}', t - r/c + \mathbf{r}\cdot\mathbf{r}'/cr)}{r - (\mathbf{r}\cdot\mathbf{r}')/r}dV' \\ &= \frac{\mu_0}{4\pi r}\int_{V_1}\mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right)dV' + \dots\end{aligned} \tag{3.71}$$

It is an exercise to show that for a finite volume V $\int_V \mathbf{J}\,dV = d\mathbf{p}/dt$, so

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi r}\dot{\mathbf{p}}(t-r/c) \tag{3.72}$$

Next we only consider radiation fields with the spatial dependence r^{-1}. Because $\dot{\mathbf{p}}$ is a function of $(t-r/c)$, then $\partial\dot{\mathbf{p}}/\partial r = -(1/c)\ddot{\mathbf{p}}$, and

$$\mathbf{E}(\mathbf{r},t) = -\frac{\mu_0}{4\pi r}\ddot{\mathbf{p}}(t-r/c) + \frac{1}{4\pi\epsilon_0}\frac{\mathbf{r}\cdot\ddot{\mathbf{p}}(t-r/c)}{c^2 r^3}\mathbf{r} \tag{3.73}$$

An exercise is to show that the magnetic radiation field is

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t) = -\frac{\mu_0}{4\pi c r^2}\mathbf{r}\times\ddot{\mathbf{p}} \tag{3.74}$$

and that

$$\mathbf{E}(\mathbf{r},t) = -\frac{c}{r}\mathbf{r}\times\mathbf{B}(\mathbf{r},t) \tag{3.75}$$

All derivatives are evaluated at the retarded time $t - r/c$, which can be assumed identical to all particles in V_1.

The radiation field is a transverse electromagnetic wave, whose Poynting vector

$$\mathbf{S} = \frac{cB^2}{\mu_0 r}\mathbf{r} = \frac{1}{16\pi^2\epsilon_0 c^3 r^5}(\mathbf{r}\times\ddot{\mathbf{p}})^2\mathbf{r} \tag{3.76}$$

Setting z-axis parallel to $\ddot{\mathbf{p}}$ yields

$$\mathbf{S} = \frac{\ddot{\mathbf{p}}^2 \sin^2\theta}{16\pi^2\epsilon_0 c^3 r^2}\frac{\mathbf{r}}{r} \tag{3.77}$$

The maximum intensity is obtained in the perpendicular direction to $\ddot{\mathbf{p}}$. To determine the radiated power P_{rad}, consider a spherical surface at a large distance:

$$P_{rad} = \oint_{\partial V} \mathbf{S}\cdot\mathbf{n}\,da = \frac{1}{6\pi\epsilon_0}\frac{\ddot{\mathbf{p}}^2}{c^3} \tag{3.78}$$

This result shows that a set of charged particles radiates if the related dipole moment has "acceleration". It may also happen that the dipole moment is independent of time, although particles have acceleration. In such a case, it would be necessary to consider higher terms in the multipole expansion.

3.2.6 Auroral kilometric radiation

Auroral kilometric radiation (AKR) is emitted by electrons in the auroral acceleration region at heights 2000-20000 km[1]. The frequency is equal to the Larmor frequency of electrons (30-600 kHz) and its wavelength is a few kilometres. The maximum radiation power during magnetic storms is about 1 GW, which is about 1 % of the power of particle precipitation into the ionosphere. The solar radiation power incident at the Earth is in turn about 10^8 GW. This is of the same order of magnitude as the infrared radiation emitted by the Earth. So AKR is not significant from the energy viewpoint.

However, the number of photons per unit time can be larger for AKR than for infrared radiation. This is remarkable, since the photon flux determines how distant objects it is possible to observe, assuming that the detector technology is sophisticated. It might be possible to observe AKR emitted up to a distance of about 100 light years (numerical argumentation is an exercise). In other words, if there are any earth-like exoplanets within the distance of 100 ly they could be observed due to their AKR emission.

There are a few problems in such an observation. First, the detector should be in space, because AKR is dissipated in the ionosphere due to its low frequency. It is evidently possible to construct such a space instrument with the present technology. Another problem is that the Sun is a more intense radiation source at AKR frequencies than the Earth, at least most of the time. Concerning exoplanets, their host stars could disturb the observation of planetary AKR. A solution is interferometry: signal must be received at two sites simultaneously. An exercise is to show that this measurement could be possible to perform in our solar system. The third difficulty may be scintillation due to interplanetary plasmas. Finally, the reader should

[1] AKR section and its exercises are based on ideas presented by Pekka Janhunen (FMI).

use her/his imagination to think of what we could infer from an observation of AKR from an Earth-like exoplanet.

3.3 Vector multipole fields

The spherical harmonic expansion is useful in electro- and magnetostatic problems having some symmetry with respect to the origin of the coordinate system. For time-varying fields the scalar spherical harmonic expansion can be generalised to a vector form. This is also more powerful than the approach based on the expansion of the vector potential (Sect. 3.1). The vector multipole approach is useful in boundary value problems with a spherical symmetry and in studies of radiation from a localised source.

3.3.1 Basic spherical wave solutions

We start with the scalar wave equation

$$\nabla^2\psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\psi(\mathbf{r},t)}{\partial t^2} = 0 \tag{3.79}$$

Fourier transform with respect to time leads to the Helmholtz equation

$$\nabla^2\psi(\mathbf{r},\omega) + k^2\psi(\mathbf{r},\omega) = 0 \tag{3.80}$$

Separation in spherical coordinates leads to the familiar expansion

$$\psi(\mathbf{r},\omega) = \sum_{l,m} f_l(r)Y_{lm}(\theta,\phi) \tag{3.81}$$

where Y_{lm} are spherical harmonics. The reader may (should) show that the general solution can be written as

$$\psi(\mathbf{r},\omega) = \sum_{l,m}(A^{(1)}_{lm}h^{(1)}_l(kr) + A^{(2)}_{lm}h^{(2)}_l(kr))Y_{lm}(\theta,\phi) \tag{3.82}$$

Here $h^{(1,2)}_l$ is the spherical Hankel function defined by

$$h^{(1,2)}_l(x) = (\frac{\pi}{2x})^{1/2}(J_{l+1/2}(x) \pm iN_{l+1/2}(x)) = j_l(x) \pm in_l(x) \tag{3.83}$$

where J and N are the Bessel and Neumann functions, and j and n are the spherical Bessel and Neumann functions, respectively. Their explicit expressions are easily computed from

$$\begin{aligned} j_l(x) &= (-x)^l(\frac{1}{x}\frac{d}{dx})^l\frac{\sin x}{x} \\ n_l(x) &= -(-x)^l(\frac{1}{x}\frac{d}{dx})^l\frac{\cos x}{x} \end{aligned} \tag{3.84}$$

For other useful formulas, see *Jackson* or *Arfken and Weber*, for example.

Next we consider more closely the spherical harmonics, which satisfy the equation

$$-(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2})Y_{lm} = l(l+1)Y_{lm} \tag{3.85}$$

It is convenient to define an operator

$$\mathbf{L} = \frac{1}{i}\mathbf{r}\times\nabla \tag{3.86}$$

Readers familiar with quantum mechanics may recognize that this is basically the orbital angular momentum operator. Its components can be written as

$$\begin{aligned} L_+ &= L_x + iL_y = e^{i\phi}(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}) \\ L_- &= L_x - iL_y = e^{-i\phi}(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}) \\ L_z &= -i\frac{\partial}{\partial\phi} \end{aligned} \tag{3.87}$$

The operator $\mathbf{L}$ affects only angular variables and is independent of r. A straightforward calculation shows that $\mathbf{L}\cdot\mathbf{L} = L^2 = L_x^2 + L_y^2 + L_z^2$ is the operator on the left hand side of Eq. 3.85:

$$L^2 Y_{lm} = l(l+1)Y_{lm} \tag{3.88}$$

The proof of the following results is an exercise:

$$\begin{aligned} \mathbf{r}\cdot\mathbf{L} &= 0 \\ L_+Y_{lm} &= \sqrt{(l-m)(l+m+1)}Y_{l,m+1} \\ L_-Y_{lm} &= \sqrt{(l+m)(l-m+1)}Y_{l,m-1} \\ L_zY_{lm} &= mY_{lm} \\ L^2\mathbf{L} &= \mathbf{L}L^2 \\ \mathbf{L}\times\mathbf{L} &= i\mathbf{L} \\ L_j\nabla^2 &= \nabla^2 L_j \\ \nabla^2 &= \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{L^2}{r^2} \\ i\nabla\times\mathbf{L} &= \mathbf{r}\nabla^2 - \nabla(1 + r\frac{\partial}{\partial r}) \\ \mathbf{L}\cdot\mathbf{F} &= i\nabla\cdot(\mathbf{r}\times\mathbf{F}) \\ \mathbf{L}\cdot(\nabla\times\mathbf{F}) &= i\nabla^2(\mathbf{r}\times\mathbf{F}) - \frac{i}{r}\frac{\partial}{\partial r}(r^2\nabla\cdot\mathbf{F}) \end{aligned} \tag{3.89}$$

3.3.2 Multipole expansion of the electromagnetic fields

Next we consider the electromagnetic field in a source-free region. Assuming time-harmonic fields ($e^{-i\omega t}$) and denoting $k = \omega/c$, the magnetic field satifies the Helmholtz equation and the divergence-free condition, and the electric field is obtained from the magnetic field:

$$\begin{aligned} (\nabla^2 + k^2)\mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \mathbf{E} &= \frac{ic}{k}\nabla \times \mathbf{B} \end{aligned} \tag{3.90}$$

Alternatively,

$$\begin{aligned} (\nabla^2 + k^2)\mathbf{E} &= 0 \\ \nabla \cdot \mathbf{E} &= 0 \\ \mathbf{B} &= -\frac{i}{kc}\nabla \times \mathbf{E} \end{aligned} \tag{3.91}$$

The goal is to represent the fields as multipole expansions with explicitly separated radial and angular dependences. One possibility is to first note that for any well-behaved vector function $\mathbf{F}$,

$$\nabla^2(\mathbf{r} \cdot \mathbf{F}) = \mathbf{r} \cdot (\nabla^2 \mathbf{F}) + 2\nabla \cdot \mathbf{F} \tag{3.92}$$

Consequently, outside of the source region the scalar functions $\mathbf{r} \cdot \mathbf{B}$ and $\mathbf{r} \cdot \mathbf{E}$ satisfy the Helmholtz equation

$$(\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{B}) = 0,\ (\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{E}) = 0 \tag{3.93}$$

The general solution for these scalar functions is given by Eq. 3.82.

Now we define a magnetic multipole field of order (l, m) by

$$\begin{aligned} \mathbf{r} \cdot \mathbf{B}_{lm}^{(M)} &= \frac{l(l+1)}{kc} g_l(kr) Y_{lm}(\theta, \phi) \\ \mathbf{r} \cdot \mathbf{E}_{lm}^{(M)} &= 0 \end{aligned} \tag{3.94}$$

where

$$g_l(kr) = A_l^{(1)} h_l^{(1)}(kr) + A_l^{(2)} h_l^{(2)}(kr) \tag{3.95}$$

The magnetic and electric fields are related by

$$kc\mathbf{r} \cdot \mathbf{B} = \frac{1}{i}\mathbf{r} \cdot (\nabla \times \mathbf{E}) = \frac{1}{i}(\mathbf{r} \times \nabla) \cdot \mathbf{E} = \mathbf{L} \cdot \mathbf{E} \tag{3.96}$$

The electric field of the magnetic multipole must then satisfy the equation

$$\mathbf{L} \cdot \mathbf{E}_{lm}^{(M)}(r, \theta, \phi) = l(l+1) g_l(kr) Y_{lm}(\theta, \phi) \tag{3.97}$$

and $\mathbf{r}\cdot\mathbf{E}_{lm}^{(M)} = 0$. Because the operator $\mathbf{L}$ acts only on the angular variables, the radial dependence of $\mathbf{E}_{lm}^{(M)}$ is given by $g_l(kr)$. As shown in the previous subsection, the effect of $\mathbf{L}$ on Y_{lm} is to raise or lower m, but not to modify l at all. Remembering that $L^2 Y_{lm} = l(l+1)Y_{lm}$, we see rather easily that the electric field must be

$$\mathbf{E}_{lm}^{(M)}(r,\theta,\phi) = g_l(kr)\mathbf{L}Y_{lm}(\theta,\phi) \tag{3.98}$$

Since

$$\mathbf{B}_{lm}^{(M)} = -\frac{i}{kc}\nabla\times\mathbf{E}_{lm}^{(M)} \tag{3.99}$$

the fields of a magnetic multipole of order (l,m) are now specified apart from the coefficients $A_l^{(1,2)}$.

In the same way, we define an electric multipole of order (l,m) by

$$\begin{aligned}\mathbf{r}\cdot\mathbf{E}_{lm}^{(E)} &= -\frac{l(l+1)c}{k}f_l(kr)Y_{lm}(\theta,\phi)\\ \mathbf{r}\cdot\mathbf{B}_{lm}^{(E)} &= 0\end{aligned} \tag{3.100}$$

The electric multipole fields are

$$\begin{aligned}\mathbf{B}_{lm}^{(E)}(r,\theta,\phi) &= f_l(kr)\mathbf{L}Y_{lm}(\theta,\phi)\\ \mathbf{E}_{lm}^{(E)} &= \frac{ic}{k}\nabla\times\mathbf{B}_{lm}^{(E)}\end{aligned} \tag{3.101}$$

The radial function $f_l(kr)$ has a similar expression to $g_l(kr)$.

We are now ready to write the general solution of the Maxwell equations in the source-free region. For more convenient notations, we define a normalized form of the vector spherical harmonic:

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\phi) \tag{3.102}$$

It has the following orthogonality properties:

$$\begin{aligned}\int \mathbf{X}_{l'm'}^*\cdot\mathbf{X}_{lm}\,d\Omega &= \delta_{ll'}\delta_{mm'}\\ \int \mathbf{X}_{l'm'}^*\cdot(\mathbf{r}\times\mathbf{X}_{lm})\,d\Omega &= 0\end{aligned} \tag{3.103}$$

Consequently, the fields are

$$\begin{aligned}\mathbf{B} &= \sum_{lm}\left[a_E(l,m)f_l(kr)\mathbf{X}_{lm} - \frac{i}{kc}a_M(l,m)\nabla\times(g_l(kr)\mathbf{X}_{lm})\right]\\ \mathbf{E} &= \sum_{lm}\left[\frac{ic}{k}a_E(l,m)\nabla\times(f_l(kr)\mathbf{X}_{lm}) + a_M(l,m)g_l(kr)\mathbf{X}_{lm}\right]\end{aligned} \tag{3.104}$$

The unknown coefficients are obtained from

$$\begin{aligned} a_M(l,m)g_l(kr) &= \frac{k}{\sqrt{l(l+1)}}\int Y^*_{lm}\ \mathbf{r}\cdot\mathbf{B}\ d\Omega \\ a_E(l,m)f_l(kr) &= -\frac{k}{\sqrt{l(l+1)}}\int Y^*_{lm}\ \mathbf{r}\cdot\mathbf{E}\ d\Omega \end{aligned} \tag{3.105}$$

Thus, we only have to know $\mathbf{r}\cdot\mathbf{B}$ and $\mathbf{r}\cdot\mathbf{E}$ at two different radii r_1 and r_2 in the source-free region for a complete solution, including the relative magnitudes of $A_l^{(1,2)}$.

3.3.3 Near and far zone multipole fields

To study the near zone fields ($kr \ll 1$), we note that the radial dependence is given by $h_l^{(1,2)}(kr) = j_l(kr) \pm in_l(kr)$. With a small kr, the spherical Hankel function behaves like $(kr)^{-(l+1)}$, and the magnetic field for an electric multipole (l,m) is

$$\mathbf{B}^{(E)}_{lm} \sim -\frac{k}{l}\mathbf{L}\frac{Y_{lm}}{r^{l+1}} \tag{3.106}$$

where the specific proportionality coefficient is chosen for convenience. The electric field is

$$\mathbf{E}^{(E)}_{lm} = \frac{ic}{k}\nabla\times\mathbf{B}^{(E)}_{lm} \sim -\frac{i}{l}\nabla\times\mathbf{L}(\frac{Y_{lm}}{r^{l+1}}) \tag{3.107}$$

Using an operator identity, the electric field takes the form

$$\mathbf{E}^{(E)}_{lm} \sim -(\mathbf{r}\nabla^2 - \nabla(1 + r\frac{\partial}{\partial r}))\frac{Y_{lm}}{lr^{l+1}} \tag{3.108}$$

The first term vanishes, since Y_{lm}/r^{l+1} is a solution of the Laplace equation. It follows that

$$\mathbf{E}^{(E)}_{lm} \sim -\nabla(\frac{Y_{lm}}{r^{l+1}}) \tag{3.109}$$

From the basic course of electrodynamics we remember that this is exactly the electrostatic multipole field. The magnetic multipole fields are treated in the same way just by interchanging $\mathbf{E}^{(E)} \to -\mathbf{B}^{(M)}, \mathbf{B}^{(E)} \to \mathbf{E}^{(M)}$.

For the far zone ($kr \gg 1$), we investigate outgoing waves from a localized source, corresponding to the radial dependence given by $h_l^{(1)}(kr)$. Its asymptotic behaviour implies that the magnetic field of an electric multipole is

$$\mathbf{B}^{(E)}_{lm} \sim (-i)^{l+1}\frac{e^{ikr}}{kr}\mathbf{L}Y_{lm} \tag{3.110}$$

and the electric field is

$$\mathbf{E}^{(E)}_{lm} \sim \frac{(-i)^l c}{k^2}(\nabla(\frac{e^{ikr}}{r})\times\mathbf{L}Y_{lm} + \frac{e^{ikr}}{r}\nabla\times\mathbf{L}Y_{lm}) \tag{3.111}$$

Keeping only the leading terms and using the same operator identity as for the near zone calculation, this becomes

$$\mathbf{E}_{lm}^{(E)} \sim -(-i)^{l+1}\frac{ce^{ikr}}{kr}(\mathbf{n}\times\mathbf{L}Y_{lm} - \frac{1}{k}(\mathbf{r}\nabla^2 - \nabla)Y_{lm}) \tag{3.112}$$

where $\mathbf{n} = \mathbf{r}/r$. The second term is clearly some angular function times $1/r$, so it is ignorable. Not surprisingly, the radiation zone fields have an exact relationship

$$\mathbf{E}_{lm}^{(E)} = c\mathbf{B}_{lm}^{(E)} \times \mathbf{n} \tag{3.113}$$

Again, the magnetic multipole fields are obtained by a similar interchange like with the near zone case.

3.3.4 Energy of multipole radiation

We start the energy consideration by studying only a linear superposition of electric multipoles (l, m) with a varying m but a fixed l. Then the outgoing fields are

$$\begin{aligned} \mathbf{B}_l &= \sum_m a_E(l,m)\mathbf{X}_{lm}h_l^{(1)}(kr) \\ \mathbf{E}_l &= \frac{ic}{k}\nabla\times\mathbf{B}_l \end{aligned} \tag{3.114}$$

The time-averaged energy density for time-harmonic fields is

$$u = \frac{\epsilon_0}{4}(\mathbf{E}\cdot\mathbf{E}^* + c^2\mathbf{B}\cdot\mathbf{B}^*) \tag{3.115}$$

In the radiation zone $\mathbf{E} = c\mathbf{B}\times\mathbf{n}$, so the energy dU in a spherical shell $[r, r+dr]$ is

$$dU = \frac{2\pi\epsilon_0 c^2 dr}{k^2}\sum_{m,m'} a_E^*(l,m')a_E(l,m)\int_{(4\pi)}\mathbf{X}_{lm'}^*\cdot\mathbf{X}_{lm}\,d\Omega \tag{3.116}$$

where the asymptotic expression of the spherical Hankel function is applied. Using the orthogonality integral of vector spherical harmonics, this yields

$$\frac{dU}{dr} = \frac{2\pi\epsilon_0 c^2}{k^2}\sum_m |a_E(l,m)|^2 \tag{3.117}$$

For a general superposition of electric and magnetic multipoles the result is

$$\frac{dU}{dr} = \frac{2\pi\epsilon_0 c^2}{k^2}\sum_{l,m}(|a_E(l,m)|^2 + |a_M(l,m)|^2) \tag{3.118}$$

Next we discuss the power of multipole radiation. At the limit $kr \gg 1$ the fields are

$$\begin{aligned} \mathbf{B} &= \frac{e^{ikr}}{kr}\sum_{lm}(-i)^{l+1}(a_E(l,m)\mathbf{X}_{lm} + a_M(l,m)\mathbf{n}\times\mathbf{X}_{lm}) \\ \mathbf{E} &= c\mathbf{B}\times\mathbf{n} \end{aligned} \tag{3.119}$$

The average power radiated per unit solid angle is calculated from Poynting's vector yielding

$$\frac{dP}{d\Omega} = \frac{c}{2\mu_0 k^2}|\sum_{lm}(-i)^{l+1}(a_E(l,m)\mathbf{X}_{lm}\times\mathbf{n} + a_M(l,m)\mathbf{X}_{lm})|^2 \tag{3.120}$$

The electric and magnetic multipoles of a given (l,m) have the same angular dependence, but perpendicular polarizations. So the multipole order can be determined by measuring the angular distribution of radiated power. To distinguish between the electric and magnetic nature of the radiating source, polarization must be detected. The angular power distribution of a single multipole is

$$\frac{dP(l,m)}{d\Omega} = \frac{c}{2\mu_0 k^2}|a(l,m)\mathbf{X}_{lm}|^2 \tag{3.121}$$

3.3.5 Multipole moments

The next task is to relate the multipole expressions directly to the source terms. We assume that the charge density $\rho(\mathbf{r})e^{-i\omega t}$ and current density $\mathbf{J}(\mathbf{r})e^{-i\omega t}$ are known. We could also consider the intrinsic magnetization as a source term, but we neglect it here for brevity (see *Jackson* for a complete discussion).

Again, we start with the Maxwell equations

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \rho/\epsilon_0 \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{E} - ikc\mathbf{B} &= 0 \\ \nabla\times\mathbf{B} + (ik/c)\mathbf{E} &= \mu_0\mathbf{J} \end{aligned} \tag{3.122}$$

Defining $\mathbf{E}' = \mathbf{E} + i\mathbf{J}/(\omega\epsilon_0)$ and using the equation of the current continuity, we obtain

$$\begin{aligned} \nabla\cdot\mathbf{E}' &= 0 \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{E}' - ikc\mathbf{B} &= i\nabla\times\mathbf{J}/(\omega\epsilon_0) \\ \nabla\times\mathbf{B} + (ik/c)\mathbf{E}' &= 0 \end{aligned} \tag{3.123}$$

If we had included magnetization, its curl would appear in the right side of the last equation. Note that $\mathbf{E}' = \mathbf{E}$ outside of sources.

The inhomogeneous Helmholtz wave equations follow from the curl equations:

$$\begin{aligned}(\nabla^2 + k^2)\mathbf{B} &= -\mu_0 \nabla \times \mathbf{J} \\ (\nabla^2 + k^2)\mathbf{E}' &= -i\nabla \times (\nabla \times \mathbf{J})/(\epsilon_0 ck) \end{aligned} \tag{3.124}$$

We use the same trick as earlier and take the scalar product with vector $\mathbf{r}$, and use the vector identity $\mathbf{r} \cdot (\nabla \times \mathbf{F}) = (\mathbf{r} \times \nabla) \cdot \mathbf{F} = -i\mathbf{L} \cdot \mathbf{F}$. This yields scalar equations

$$\begin{aligned}(\nabla^2 + k^2)\mathbf{r} \cdot \mathbf{B} &= -i\mu_0 \mathbf{L} \cdot \mathbf{J} \\ (\nabla^2 + k^2)\mathbf{r} \cdot \mathbf{E}' &= \mathbf{L} \cdot (\nabla \times \mathbf{J})/(\epsilon_0 ck) \end{aligned} \tag{3.125}$$

These equations can be solved using the method of Green's functions (see the course of electrodynamics). Since we are interested in outgoing waves, the solution is

$$\begin{aligned}\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) &= \frac{i\mu_0}{4\pi} \int \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \, \mathbf{L}' \cdot \mathbf{J}(\mathbf{r}') \, d^3\mathbf{r}' \\ \mathbf{r} \cdot \mathbf{E}'(\mathbf{r}) &= -\frac{1}{4\pi\epsilon_0 ck} \int \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \, \mathbf{L}' \cdot \nabla' \times \mathbf{J}(\mathbf{r}') \, d^3\mathbf{r}' \end{aligned} \tag{3.126}$$

The multipole coefficients a_E, a_M are obtained from

$$\begin{aligned}a_M(l,m) g_l(kr) &= \frac{k}{\sqrt{l(l+1)}} \int Y^*_{lm} \, \mathbf{r} \cdot \mathbf{B} \, d\Omega \\ a_E(l,m) f_l(kr) &= -\frac{k}{\sqrt{l(l+1)}} \int Y^*_{lm} \, \mathbf{r} \cdot \mathbf{E}' \, d\Omega \end{aligned} \tag{3.127}$$

as derived in Sect. 3.3.2. The radial dependence must be $f_l(kr) = g_l(kr) = h_l^{(1)}(kr)$, since this corresponds to outgoing radiation. Next, we need Eq. 3.5:

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{l=0}^{\infty} j_l(kr_<) h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi) \tag{3.128}$$

where now $r_< = r', r_> = r$, because we study the region outside of the source. It follows that

$$\int d\Omega \; Y^*_{lm}(\theta, \phi) \; \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = 4\pi ik h_l^{(1)}(kr) j_l(kr') Y^*_{lm}(\theta', \phi') \tag{3.129}$$

Consequently, the multipole coefficients are

$$\begin{aligned} a_M(l,m) &= -\frac{\mu_0 k^2}{\sqrt{l(l+1)}} \int j_l(kr) Y^*_{lm}(\theta,\phi)\ \mathbf{L}\cdot\mathbf{J}(\mathbf{r})\ d^3\mathbf{r} \\ a_E(l,m) &= \frac{ik}{\epsilon_0 c\sqrt{l(l+1)}} \int j_l(kr) Y^*_{lm}(\theta,\phi)\ \mathbf{L}\cdot\nabla\times\mathbf{J}(\mathbf{r})\ d^3\mathbf{r} \end{aligned} \tag{3.130}$$

A centre-fed linear antenna (Sect. 3.2.3) provides an illustrating concrete exercise.