

# Chapter 4

## Scattering

The previous chapter dealt with fields due to charge and current distributions in an otherwise empty space. Now we will turn to a more general situation with some material under the influence of electromagnetic fields. We will study scattering by small objects whose sizes are much smaller than the wavelength of the incoming field. Then a multipole approach is convenient. We will also introduce a method to model a medium with inclusions of a different material. We limit that consideration to an electrostatic (or magnetostatic) case, but results are also applicable in scattering problems with certain limitations.

### 4.1 Scattering at long wavelengths

#### 4.1.1 Dipole approximation

Consider a piece of material whose dimensions are much smaller than the wavelength of the incoming electromagnetic plane wave. The assumption that the incoming field is a plane wave is clearly acceptable. Let  $\mathbf{n}_0$  be the direction of the incident (primary) wave and  $\mathbf{u}_0$  the direction of polarization. Then

$$\begin{aligned}\mathbf{E}_{inc} &= \mathbf{u}_0 E_0 e^{ik\mathbf{n}_0 \cdot \mathbf{r}} \\ \mathbf{B}_{inc} &= \mathbf{u}_0 \times \mathbf{E}_{inc}/c\end{aligned}\tag{4.1}$$

where  $k = \omega/c$  and the time-dependence is the usual  $e^{-i\omega t}$ . As shown in the previous chapter, the scattered (secondary) fields at large distances (radiation zone) are

$$\begin{aligned}\mathbf{E}_{sc} &= \frac{\mu_0 c k^2 e^{ikr}}{4\pi r} (c(\mathbf{n} \times \mathbf{p}) \times \mathbf{n} - \mathbf{n} \times \mathbf{m}) \\ \mathbf{B}_{sc} &= \mathbf{n} \times \mathbf{E}_{sc}/c\end{aligned}\tag{4.2}$$

where  $\mathbf{n}$  is the unit vector from the scatterer to the observation point and  $r$  is the distance between the scatterer and observer.

A widely used quantity is the differential scattering cross section. It is the power radiated in the direction  $\mathbf{n}$  with polarization  $\mathbf{u}$ , per unit solid angle, per unit incident flux in the direction  $\mathbf{n}_0$  with polarization  $\mathbf{u}_0$ :

$$\frac{d\sigma}{d\Omega}(\mathbf{n}, \mathbf{u}; \mathbf{n}_0, \mathbf{u}_0) = r^2 \frac{|\mathbf{u}^* \cdot \mathbf{E}_{sc}|^2}{|\mathbf{u}_0^* \cdot \mathbf{E}_{inc}|^2} \quad (4.3)$$

Note that the complex conjugate of the polarization vector is important. Using the dipole approximation, the differential cross section takes the form

$$\frac{d\sigma}{d\Omega}(\mathbf{n}, \mathbf{u}; \mathbf{n}_0, \mathbf{u}_0) = \frac{\mu_0^2 c^2 k^4}{16\pi^2 E_0^2} |\mathbf{u}_0^* \cdot \mathbf{p} + (\mathbf{n} \times \mathbf{u}_0^*) \cdot \mathbf{m}|^2 \quad (4.4)$$

The dependence of the cross section on  $\mathbf{n}_0$  and  $\mathbf{u}_0$  is implicitly contained in the dipole moment vectors  $\mathbf{p}$  and  $\mathbf{m}$ , since they depend on the incident wave. The cross section is proportional to  $k^4$  or equivalently to  $1/\lambda^4$ , a result known as Rayleigh's law. This is a universal characteristic of the scattering of long wavelength radiation by any finite system, except for a special case that both dipole moments happen to vanish. For the visible light the blue colour has the shortest wavelength, so Rayleigh's law gives the basic explanation for the colour of the sky.

### 4.1.2 Dielectric spherical scatterer

As an example of dipole scattering, consider a small neutral dielectric sphere of radius  $a$  and permittivity  $\epsilon$ . We know from electrostatics that the induced dipole moment is

$$\mathbf{p} = 4\pi\epsilon_0 a^3 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \mathbf{E}_{inc} \quad (4.5)$$

Due to the absence of free charges, there is no magnetic dipole moment. The differential scattering cross section is now

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left| \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right|^2 |\mathbf{u}^* \cdot \mathbf{u}_0|^2 \quad (4.6)$$

The scattered radiation is linearly polarised in the plane defined by the dipole moment direction ( $\mathbf{u}_0$ ) and the unit vector  $\mathbf{n}$ .

The incident radiation is typically unpolarised. Then it is interesting to study the angular distribution of scattered radiation of a definite state of linear polarization( $\mathbf{u}$ ). We define the scattering plane by vectors  $\mathbf{n}_0$  and  $\mathbf{n}$  to be the  $xz$ -plane. The polarization vectors  $\mathbf{u}_0^{(1)}$  and  $\mathbf{u}^{(1)}$  are in this plane, and

$\mathbf{u}_0^{(2)} = \mathbf{u}^{(2)}$  is perpendicular to it. The differential cross sections for scattering with the polarizations  $\mathbf{u}^{(1)}$  and  $\mathbf{u}^{(2)}$  averaged over initial polarizations are

$$\begin{aligned}\frac{d\sigma_{\parallel}}{d\Omega} &= \frac{k^4 a^6}{2} \left| \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right|^2 \cos^2 \theta \\ \frac{d\sigma_{\perp}}{d\Omega} &= \frac{k^4 a^6}{2} \left| \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right|^2\end{aligned}\quad (4.7)$$

where  $\theta$  is the angle between  $\mathbf{n}_0$  and  $\mathbf{n}$  (as well between  $\mathbf{u}_0$  and  $\mathbf{u}$ ). The subscripts  $\parallel$  and  $\perp$  denote polarization parallel and perpendicular to the scattering plane, respectively.

The polarization  $\Pi(\theta)$  of the scattered radiation is defined by

$$\Pi(\theta) = \frac{d\sigma_{\perp}/d\Omega - d\sigma_{\parallel}/d\Omega}{d\sigma_{\perp}/d\Omega + d\sigma_{\parallel}/d\Omega} \quad (4.8)$$

which for the present example is

$$\Pi(\theta) = \frac{\sin^2 \theta}{1 + \cos^2 \theta} \quad (4.9)$$

The differential cross section is the sum of parallel and perpendicular contributions:

$$\frac{d\sigma}{d\Omega} = \frac{k^4 a^6}{2} \left| \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right|^2 (1 + \cos^2 \theta) \quad (4.10)$$

The total cross section is

$$\sigma = \int_{(4\pi)} \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi k^4 a^6}{3} \left| \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right|^2 \quad (4.11)$$

Exercise: plot  $d\sigma/d\Omega$  and  $\Pi(\theta)$ .

### 4.1.3 Perfectly conducting spherical scatterer

In the case of a perfectly conducting small sphere, both electric and magnetic dipoles are present. Induced dipole moments familiar from electro- and magnetostatics are

$$\mathbf{p} = 4\pi\epsilon_0 a^3 \mathbf{E}_{inc} \quad (4.12)$$

and

$$\mathbf{m} = -\frac{2\pi a^3}{\mu_0} \mathbf{B}_{inc} \quad (4.13)$$

An exercise is to determine the polarization and the total cross section.

## 4.2 Maxwell Garnett mixing formula

The material parameters in the Maxwell equations are macroscopic quantities. Therefore, the electromagnetic fields solved from the equations are some kind of averages of the microscopic field distributions. For example, when we study radio wave propagation in the air we do not solve the fields acting between single electrons and protons nor even between various molecules, but we calculate the average fields in the size scale of  $10^{20}$  atoms. However, to be able to get good macroscopic approximations, we have to take into account what is happening in the microscopic level. To model realistically air as a dielectric material, it is necessary to consider the different scales of microscopic inhomogeneities. Air consists of many different particles of different sizes. It is a mixture of gases, liquids and solids. In case of rain, condensed water phase in the form of drops gives a considerable contribution to the permittivity.

Mixing theories aim at predicting the effective response of mixtures without having to calculate the microscopic fields. The goal of the theories is to provide a method to determine the effective material parameters (permittivity, permeability, conductivity) that can be used in the Maxwell equations. The following presentation is based on electrostatics. However, results are applicable also to time-dependent cases assuming that sizes of scatterers are much smaller than the wavelength.

The oldest mixing formula is by Maxwell Garnett<sup>1</sup>. To derive this result, we start by defining the molecular polarizability  $\alpha$  of a single molecule:

$$\mathbf{p}_m = \alpha \epsilon_0 \mathbf{E}_m \quad (4.14)$$

where  $\mathbf{p}_m$  is the induced dipole moment and  $\mathbf{E}_m$  is the polarizing electric field at the location of the molecule. For simplicity, we assume that there is no permanent polarization. We also assume that the polarizability is a scalar, since we will restrict the treatment to uniform spherical inclusions.

The polarizing field is produced by all external sources and by polarized molecules except the one under study. We can imagine that we remove a small cavity around the molecule. Now  $\mathbf{E}_m$  can be expressed as

$$\mathbf{E}_m = \mathbf{E} + \mathbf{E}_p + \mathbf{E}_{near} \quad (4.15)$$

Here  $\mathbf{E}$  is the average field in the whole body,  $\mathbf{E}_p$  the field due to polarization charges on the surface of the cavity and  $\mathbf{E}_{near}$  is due to all other molecules inside the cavity. We assume that the structure is regular enough, like a cubical grid, or that molecules are randomly distributed. Then  $\mathbf{E}_{near}$  is zero.

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<sup>1</sup>Maxwell Garnett, J.C., Colours in metal glasses and metal films, Transactions of the Royal Society, CCIII, 385-420, 1904.

Determination of  $\mathbf{E}_p$  is a little tricky, since the result depends on the shape of the cavity. Since we will later study spherical inclusions, the cavity is also assumed to be spherical. Now we can use the result that the electric field in a uniformly polarized sphere is  $-\mathbf{P}/(3\epsilon_0)$ . Consequently, in the cavity  $\mathbf{E}_m = \mathbf{E} + \mathbf{P}/(3\epsilon_0)$ .

If the number density of molecules is  $n$  then the polarization is according to its definition  $\mathbf{P} = n\mathbf{p}_m$  yielding

$$\mathbf{P} = n\alpha\epsilon_0(\mathbf{E} + \mathbf{P}/(3\epsilon_0)) \quad (4.16)$$

On the other hand  $\mathbf{P} = (\epsilon_r - 1)\epsilon_0\mathbf{E}$ , which results in the Clausius and Mossotti equation

$$\alpha = \frac{3(\epsilon_r - 1)}{n(\epsilon_r + 2)} \quad (4.17)$$

Although we have so far considered a molecule, it could as well be a macroscopic particle of a spherical shape.

Next we investigate a collection of inclusions which are homogeneous spheres having a permittivity  $\epsilon$ . The polarizability of a single sphere follows directly from the solution of the Laplace equation with a uniform background field. The distortion of the sphere outside its surface is exactly equal to that by a dipole placed at the centre of the sphere. Using this familiar result, we see that

$$\alpha = 3V \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \quad (4.18)$$

where  $V$  is the volume of the sphere and the background material is like vacuum.

Homogenization of the mixture means that the two expressions of the polarizability are set equal. This yields the effective permittivity of the material composed of spherical inclusions:

$$\epsilon_{eff} = \epsilon_r\epsilon_0 = \epsilon_0 + 3f\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0 - f(\epsilon - \epsilon_0)} \quad (4.19)$$

where  $f = nV$  is the volume fraction of spheres ( $0 \leq f \leq 1$ ). It is obvious that for a host medium of permittivity  $\epsilon_h$  with inclusions of permittivity  $\epsilon_i$  the effective permittivity is

$$\epsilon_{eff} = \epsilon_h + 3f\epsilon_h \frac{\epsilon_i - \epsilon_h}{\epsilon_i + 2\epsilon_h - f(\epsilon_i - \epsilon_h)} \quad (4.20)$$

This is the basic form of the Maxwell Garnett mixing formula. It is easy to check that this gives meaningful results for the special cases  $\epsilon_i = \epsilon_h$ ,  $f = 0$ , and  $f = 1$ . A few curves are shown in Fig. 4.2. It is evident that the validity of the formula is best at low volume fractions, since its derivation is based

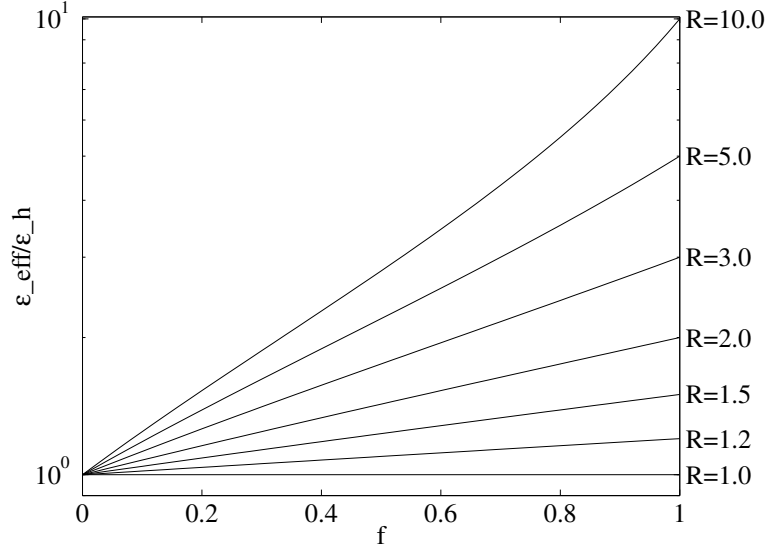


Figure 4.1: Effective permittivity  $\epsilon_{eff}$  according to the Maxwell Garnett mixing formula. The permittivity of the host material is  $\epsilon_h$  and of the inclusions  $R\epsilon_h$ . The volume fraction of the inclusions is  $f$ . The vertical scale is logarithmic.

on the analytical treatment of one sphere in an infinite space. More complicated situations are studied by Avelin (2003)<sup>2</sup> and Kärkkäinen (2003)<sup>3</sup>, for example.

As a short note, we mention the more general definition of the polarizability of a general object is

$$\mathbf{p}_m = \epsilon_0 \bar{\bar{\alpha}} \cdot \mathbf{E}_m \quad (4.21)$$

In most cases  $\bar{\bar{\alpha}}$  is a tensor, like for an ellipsoid. It is a scalar only for some special objects, including the five regular polyhedra: tetrahedron, hexahedron (cube), octahedron, dodecahedron and icosahedron (see Avelin, 2003).

<sup>2</sup>Avelin, J., Polarizability analysis of canonical dielectric and bi-anisotropic scatterers, HUT/Electromagnetics Laboratory, Ph.D. thesis, 2003.

<sup>3</sup>Kärkkäinen, K., On the finite-difference modelling of electromagnetic problems in structured lattices, HUT/Electromagnetics Laboratory, Ph.D. thesis, 2003.