Chapter 6

Wave guides and resonant cavities

The course of electrodynamics introduces ideal models of wave guides with perfectly conducting walls. We start by a short discussion about power losses in wave guides whose walls have a finite conductivity. Next, we will consider dielectric wave guides, and finally an idealised resonant cavity with spherical surfaces.

To refresh memory, we outline briefly how to handle cylindrical wave guides in general. Considering propagating waves parallel to the cylinder axis (z), we may assume a dependence $e^{i(kz-\omega t)}$. The fields have the form

$$\mathbf{E}(x,y,z) = \mathbf{E}(x,y)e^{i(kz-\omega t)} \ , \ \mathbf{B}(x,y,z) = \mathbf{B}(x,y)e^{i(kz-\omega t)} \tag{6.1}$$

from which it follows that

$$(\nabla_t^2 + \frac{\omega^2}{c^2} - k^2)\mathbf{E} = 0 \ , \ (\nabla_t^2 + \frac{\omega^2}{c^2} - k^2)\mathbf{B} = 0 \tag{6.2}$$

where

$$\nabla_t = \nabla - \mathbf{e}_z\frac{\partial}{\partial z} \ , \ \nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2} \tag{6.3}$$

We decompose the fields into parallel and transverse parts:

$$\mathbf{E} = \mathbf{E}_z + \mathbf{E}_t \tag{6.4}$$

where

$$
\begin{aligned}
\mathbf{E}_z &= (\mathbf{E} \cdot \mathbf{e}_z)\mathbf{e}_z \\
\mathbf{E}_t &= (\mathbf{e}_z \times \mathbf{E}) \times \mathbf{e}_z
\end{aligned}
\tag{6.5}
$$

Now the Maxwell equations are

$$\nabla_t \cdot \mathbf{E}_t = -\frac{\partial E_z}{\partial z} = -ikE_z \qquad (6.6)$$

$$\nabla_t \cdot \mathbf{B}_t = -\frac{\partial B_z}{\partial z} = -ikB_z \qquad (6.7)$$

$$\mathbf{e}_z \cdot (\nabla_t \times \mathbf{E}_t) = i\omega B_z \qquad (6.8)$$

$$\nabla_t E_z - \frac{\partial \mathbf{E}_t}{\partial z} = \nabla_t E_z - ik\mathbf{E}_t = i\omega \mathbf{e}_z \times \mathbf{B}_t \qquad (6.9)$$

$$\mathbf{e}_z \cdot (\nabla_t \times \mathbf{B}_t) = -\frac{i\omega}{c^2} E_z \qquad (6.10)$$

$$\nabla_t B_z - \frac{\partial \mathbf{B}_t}{\partial z} = \nabla_t B_z - ik\mathbf{B}_t = -i\frac{\omega}{c^2}\mathbf{e}_z \times \mathbf{E}_t \qquad (6.11)$$

If B_z ja E_z are known then the transverse components can be solved from 6.9 ja 6.11.

6.1   Fields at the surface and inside a conductor

In the idealized case of a perfect conductor, both the electric and the magnetic fields vanish inside the conductor. This is due to surface charge and current distributions, which react immediately to the changes of external fields to cancel fields inside the conductor. Just outside the electric field has only a normal component and the magnetic field only a tangential component. The former is proportional to the surface charge density and the latter to the surface current density.

Concerning real metallic wave guides, their walls have finite conductivities. It follows that there are power losses when the electromagnetic signal propagates along the guide. The purpose of the following discussion is to derive a scheme to estimate these losses. We could solve the problem analytically in case of simple structures, but we prefer to handle a general situation.

As we have learned from the solution of the half-space problem (Sect. 5), the skin depth describes the attenuation scale of the fields in a uniformly conducting medium. We assume that the skin depth is clearly smaller than, for example, the wall thickness of the wave guide. Then we know that there is only a thin transition layer within which the fields practically go to zero. So we can first solve the fields inside an ideal wave guide and use them to estimate losses in the transition layer.

The Maxwell curl equations, neglecting the displacement current, are inside the conductor

$$
\begin{aligned}
\mathbf{E_c} &= \nabla \times \mathbf{H}_c/\sigma \\
\mathbf{H_c} &= \nabla \times \mathbf{E}_c/(i\omega\mu)
\end{aligned}
\tag{6.12}
$$

Due to the rapid attenuation of the fields in the conductor, we can assume gradients normal to the surface much larger than parallel to it. In other words, gradient is nearly equal to its normal component:

$$
\nabla \approx -\mathbf{n}\frac{\partial}{\partial z}
\tag{6.13}
$$

where $\mathbf{n}$ is the unit normal vector outward from the conductor and z is the normal coordinate inward into the conductor. So the curl equations are

$$
\begin{aligned}
\mathbf{E_c} &\approx -\frac{1}{\sigma}\mathbf{n}\times\frac{\partial\mathbf{H}_c}{\partial z} \\
\mathbf{H_c} &\approx \frac{i}{\omega\mu}\mathbf{n}\times\frac{\partial\mathbf{E}_c}{\partial z}
\end{aligned}
\tag{6.14}
$$

It follows for the magnetic field that

$$
\begin{aligned}
\frac{\partial^2}{\partial z^2}(\mathbf{n}\times\mathbf{H}_c) + \frac{2i}{\delta^2}(\mathbf{n}\times\mathbf{H}_c) &\approx 0 \\
\mathbf{n}\cdot\mathbf{H_c} &\approx 0
\end{aligned}
\tag{6.15}
$$

where $\delta = \sqrt{2/(\omega\mu\sigma)}$ is the skin depth.

The magnetic field inside the conductor is parallel to the surface, and it decays exponentially:

$$
\mathbf{H}_c = \mathbf{H}_\| e^{-z/\delta}e^{iz/\delta}
\tag{6.16}
$$

where $\mathbf{H}_\|$ is the tangential field outside the surface. The electric field inside the conductor is

$$
\mathbf{E}_c \approx (1-i)\sqrt{\frac{\omega\mu}{2\sigma}}\,\mathbf{n}\times\mathbf{H}_\| e^{-z/\delta}e^{iz/\delta}
\tag{6.17}
$$

Since this is tangential, the continuity condition implies that there is a small tangential electric field just outside the surface too. An easy exercise is to show that $|E_c/(cB_\|)| \ll 1$.

Due to the existence of tangential electric and magnetic fields just outside the surface, there is a power flow into the conductor. The time-averaged power absorbed per unit area is

$$
\frac{dP}{dA} = \frac{1}{2}Re(\mathbf{n}\cdot\mathbf{E}\times\mathbf{H}^*) = \frac{\omega\mu\delta}{4}|\mathbf{H}_\||^2
\tag{6.18}
$$

The reader may check that this is equal to the Ohmic losses inside the conductor. Because the current is confined in a thin layer, it can be approximated by an effective surface current density

$$\mathbf{K}_{eff} \approx \mathbf{n} \times \mathbf{H}_{\parallel} \tag{6.19}$$

So the estimated power loss can be determined from

$$\frac{dP}{dA} \approx \frac{1}{2\sigma\delta} \left|\mathbf{K}_{eff}\right|^2 \tag{6.20}$$

6.2   Dielectric wave guides

The basic model of an optical wave guide is a dielectric straight cylinder surrounded by air. Fields inside (1) and outside (0) of the cylinder satisfy equations

$$\begin{aligned}
(\nabla_t^2 + \omega^2 \mu_1 \epsilon_1 - k_1^2)\mathbf{F} &= 0 \\
(\nabla_t^2 + \omega^2 \mu_0 \epsilon_0 - k_0^2)\mathbf{F} &= 0
\end{aligned} \tag{6.21}$$

where $\mathbf{F}$ is $\mathbf{B}$ or $\mathbf{E}$ and $\nabla_t^2 = \nabla^2 - \partial^2/\partial z^2$. The boundary conditions at the surface of the cylinder are the continuity of $\mathbf{B}_n$, $\mathbf{D}_n$, $\mathbf{E}_t$ and $\mathbf{H}_t$. This implies that the wave number must be the same inside and outside, $k_0 = k_1 = k$. The quantity $\gamma^2 = \omega^2 \mu_1 \epsilon_1 - k^2$ must be positive inside to get finite fields. The outside fields must vanish rapidly at large distances so that there is no energy flow across the surface of the cylinder. So $\beta^2 = k^2 - \omega^2 \mu_0 \epsilon_0$ must be positive.

Next, we study a cylinder with a circular cross section of radius a and $\mu_1 = \mu_0$. Assuming that there is no angular dependence, the parallel components $F_z = B_z, E_z$ fulfill the Bessel equation

$$\begin{aligned}
(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \gamma^2)F_z(\rho) = 0, \ \rho < a \\
(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \beta^2)F_z(\rho) = 0, \ \rho > a
\end{aligned} \tag{6.22}$$

The physically acceptable solutions are

$$\begin{aligned}
F_z(\rho) = J_0(\gamma\rho), \ \rho \le a \\
F_z(\rho) = AK_0(\beta\rho), \ \rho > a
\end{aligned} \tag{6.23}$$

where J_0 is the Bessel function and K_0 is the modified Bessel function. The constant A will be determined by boundary conditions. A straightforward

exercise is to show that the transverse components inside are

$$
\begin{aligned}
B_\rho &= \frac{ik}{\gamma^2}\frac{\partial B_z}{\partial \rho} \\
B_\phi &= \frac{i\omega\mu_0\epsilon_1}{\gamma^2}\frac{\partial E_z}{\partial \rho} \\
E_\phi &= -\frac{\omega}{k}B_\rho \\
E_\rho &= \frac{k}{\omega\mu_0\epsilon_1}B_\phi
\end{aligned}
\tag{6.24}
$$

and outside

$$
\begin{aligned}
B_\rho &= -\frac{ik}{\beta^2}\frac{\partial B_z}{\partial \rho} \\
B_\phi &= -\frac{i\omega\mu_0\epsilon_0}{\beta^2}\frac{\partial E_z}{\partial \rho} \\
E_\phi &= -\frac{\omega}{k}B_\rho \\
E_\rho &= \frac{k}{\omega\mu_0\epsilon_0}B_\phi
\end{aligned}
\tag{6.25}
$$

Furthermore, the triplets (B_z, B_ρ, E_ϕ) (TE mode) and (E_z, E_ρ, B_ϕ) (TM mode) are independent of each other.

The fields of the TE mode inside the cylinder are

$$
B_z = J_0(\gamma\rho), \ \ B_\rho = -\frac{ik}{\gamma}J_1(\gamma\rho), \ \ E_\phi = \frac{i\omega}{\gamma}J_1(\gamma\rho)
\tag{6.26}
$$

and outside

$$
B_z = AK_0(\beta\rho), \ \ B_\rho = \frac{ikA}{\beta}K_1(\beta\rho), \ \ E_\phi = -\frac{i\omega A}{\beta}K_1(\beta\rho)
\tag{6.27}
$$

Application of the boundary conditions at $\rho = a$ yields

$$
-\frac{J_1(\gamma a)}{\gamma J_0(\gamma a)} = \frac{K_1(\beta a)}{\beta K_0(\beta a)}
\tag{6.28}
$$

This is a transcendental equation for the wave number k, taking into account the additional condition

$$
\gamma^2 + \beta^2 = \mu_0(\epsilon_1 - \epsilon_0)\omega^2
\tag{6.29}
$$

Figure 6.1 shows the graphical solution of this equation. The uniform line is the left-hand-side of Eq. 6.28 and the dashed line is its right-hand-side. The vertical uniform lines are at the zeros of $J_0(\gamma a)$, and the vertical dashed line shows the zero point of β. In this example, the zero point of β is larger than
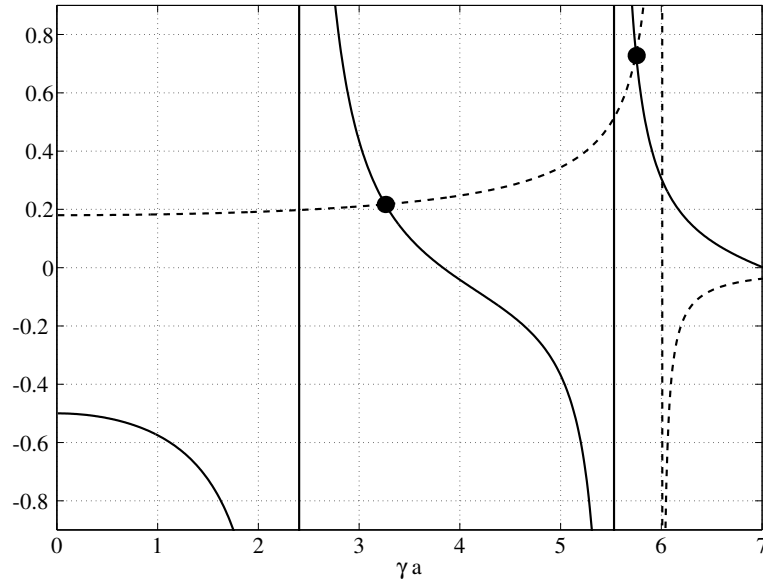
Figure 6.1: Graphical solution of Eq. 6.28. The uniform line is the left-hand-side of Eq. 6.28 and the dashed line is its right-hand-side. The vertical uniform lines are at the zeros of $J_0(\gamma a)$, and the vertical dashed line shows the zero point of β. Solutions of the equation are marked by black dots.

the two first zero points of $J_0(\gamma a)$, so there are two possible values of k. If the zero point of β were smaller than 2.405 then there would be no solution with real β. Consequently, the smallest cut-off frequency is

$$\omega_{01} \approx \frac{2.405}{a\sqrt{\mu_0(\epsilon_1 - \epsilon_0)}} \tag{6.30}$$

At this frequency $\beta^2 = 0$ and $k = \omega_{01}\sqrt{\mu_0\epsilon_0}$. An exercise is to show that immediately below ω_{01} the system is not any more a wave guide but a radially radiating antenna.

6.3   Schumann resonances

A natural resonant cavity powered by lightning exists between the earth and the ionosphere. Qualitatively, the characteristic wavelength is of the order of the Earth's radius, and the corresponding frequency is some tens of Hz. Then the skin depths in the ground and the ionosphere are small enough that, as the first approximation, the cavity can be modelled as a non-conducting volume between perfectly conducting shells (radii a and $b = a + h$).

If we only consider the lowest frequencies we can focus attention on the TM mode with no radial magnetic field. Namely, the TM modes with a radial electric field can satisfy the boundary condition of a vanishing tangential electric field at $r = a$ and $r = b$ without a significant radial variation of the fields. On the other hand, the TE modes with only a tangential electric field must have a radial variation of the order of h between the shells. It follows that the lowest frequencies of the TM and TE modes are $\omega_{TM} \sim c/a$ and $\omega_{TE} \sim c/h$.

We study only TM modes and assume further that there is no azimuthal (ϕ) dependence. Then the magnetic field can have only the ϕ component, which follows from $\nabla \cdot \mathbf{B} = 0$ and the requirement of finite fields at $\theta = 0$. Faraday's law then implies that $E_\phi = 0$ so that the only non-vanishing components are B_ϕ, E_r, E_θ.

With the harmonic time-dependence we again end at the wave equation

$$\nabla^2 \mathbf{B} + \frac{\omega^2}{c^2}\mathbf{B} = 0 \tag{6.31}$$

The ϕ component yields

$$\frac{\omega^2}{c^2}rB_\phi + \frac{\partial^2}{\partial r^2}(rB_\phi) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \ rB_\phi)\right) = 0 \tag{6.32}$$

whose angular part can be written as

$$\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \ rB_\phi)\right) = \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial(rB_\phi)}{\partial \theta}\right) - \frac{rB_\phi}{\sin^2\theta} \tag{6.33}$$

Comparison to the differential equation of the associated Legendre polynomials indicates that the field can be written as

$$B_\phi(r,\theta) = \frac{u_l}{r}P_l^1(\cos\theta) \tag{6.34}$$

where $l = 1, 2, ...$ The differential equation for u_l is then

$$\frac{d^2u_l(r)}{dr^2} + \left(\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}\right)u_l(r) = 0 \tag{6.35}$$

which is related to the spherical Bessel functions.

To apply the boundary conditions, we need the tangential electric field

$$E_\theta = -\frac{ic^2}{\omega r}\frac{\partial}{\partial r}(rB_\phi) = -\frac{ic^2}{\omega r}\frac{du_l(r)}{dr}P_l^1(\cos\theta) \tag{6.36}$$

This must vanish at $r = a, r = b$, which implies that $du_l(r)/dr$ must vanish at these shells. This leads to a transcendental equation for the characteristic frequencies, and a detailed consideration is left as an exercise. However,

rough values can be found using the relation $h/a \ll 1$. Now it is safe to replace $l(l+1)/r^2$ by $l(l+1)/a^2$. The differential equation for u_l has then trigonometric solutions $\sin qr$ and $\cos qr$, where

$$q^2 \quad = \quad \omega^2/c^2 - l(l+1)/a^2$$

The boundary condition is satisfied if $u_l(r) \sim \cos q(r-a)$ and $qh = n\pi, n = 0, 1, 2, \dots$. Only the case $n = 0$ can provide very low frequencies:

$$\omega_l \approx \sqrt{l(l+1)}\ c/a$$

These are called Schumann resonances, and the five first frequencies are $\omega_l/(2\pi) = 10.6,\ 18.3,\ 25.8,\ 33.4,\ 40.9$ Hz. The observed values are about 8, 14, 20, 26, 32 Hz.