Applications of electrodynamics, spring 2005

Exercise 3 (Thursday 10.2., return answers until 16:00 on Monday 7.2.)

1. The maximum power of the Earth's auroral kilometric radiation (AKR) is about 1 GW. The power of the infrared radiation emitted by the Earth is about 10^8 GW. The wavelengths of these radiations are about 3 km and 10 μm, respectively.
   a) Calculate the number of photons per unit time for these radiation types at the given powers and wavelengths.
   b) Assume that an exoplanet at a distance of 100 light years emits AKR as described above. How many photons would a detector of an area 100 m^2 observe in one second close to the Earth?
   c) AKR radiated by an exoplanet may be screened due to radiation by its host star. However, the planetary AKR can then be observed using interferometry. One way to perform this is to place two detectors in an orbit circulating our Sun. Determine the required distance of these instruments if the aim is to observe exoplanet AKR from a distance of 100 ly. Assume that the exoplanet orbit has a radius of 1 AU.

2. Prove the following orthogonality properties of the vector spherical harmonics:

$$\begin{aligned} \int \mathbf{X}^*_{l'm'} \cdot \mathbf{X}_{lm} \, d\Omega &= \delta_{ll'}\delta_{mm'} \\ \int \mathbf{X}^*_{l'm'} \cdot (\mathbf{r} \times \mathbf{X}_{lm}) \, d\Omega &= 0 \end{aligned}$$

3. a) The angular distribution of radiated power due to a single multipole (l, m) is

$$\frac{dP(l,m)}{d\Omega} = \frac{c}{8\pi k^2} |a(l,m)\mathbf{X}_{lm}|^2$$

   Express this in terms of spherical harmonics.
   b) Show that

$$\sum_{m=-l}^{l} |\mathbf{X}_{lm}|^2 = \frac{2l+1}{4\pi}$$

   c) Show that the total power radiated by a general source is

$$P = \frac{c}{8\pi k^2} \sum_{l,m} (|a_E(l,m)|^2 + |a_M(l,m)|^2)$$

4. Consider a thin linear antenna of length d excited by a coaxial cable across a small gap at its midpoint. The current density is

$$\mathbf{J}(\mathbf{r}) = I \sin(kd/2 - k|z|)\theta(|z| - d/2)\delta(x)\delta(y)\mathbf{e}_z$$

a) Calculate the electric multipole moments $a_E(l, m)$. Why are there no magnetic multipole moments?
b) Next consider a half-wave antenna ($kd = \pi$) and a full-wave antenna ($kd = 2\pi$). Calculate the power radiated per unit solid angle keeping only terms $l \leq 3$. Compare to the exact results derived in Sect. 3.2.3.

5. Additional challenging problem (extra points available): What an observation of AKR from an Earth-like exoplanet might indicate about conditions at that planet? You may think about this until Monday 14.2. Answers will be discussed during the 4th exercise lecture.

Acknowledgement: problems related to AKR are based on ideas by Pekka Janhunen (FMI).