Applications of electrodynamics, spring 2005

Exercise 4 (Thursday 17.2., return answers until 16:00 on Monday 14.2.)

1. Determine the polarization and the total cross section of a perfectly conducting sphere whose radius is much smaller than the wavelength of the incident electromagnetic field.

2. a) Show that the Maxwell Garnett mixing formula for metallic inclusions ($\epsilon_i \to \infty$) is

$$\epsilon_{eff} = \epsilon_h \, \frac{1+2f}{1-f}$$

b) The Maxwell Garnett formula is accurate at low volume fractions f. One attempt to improve its validity at larger f is to consider the following iterative process:
i) Start with a uniform host material of volume V_0 and permittivity ϵ_h.
ii) Add a small amount $\Delta V \ll V_0$ of metallic spheres and calculate the effective permittivity of this mixture.
iii) Consider the medium of the previous step as a uniform host material and add again inclusions of an amount ΔV. Continue this procedure several times.
Show that after N iterations

$$\epsilon_{eff} = \epsilon_h \, \prod_{n=1}^{N} \frac{1 + \frac{2\Delta V}{V_0 + n\Delta V}}{1 - \frac{\Delta V}{V_0 + n\Delta V}}$$

Show that at the limit $N \to \infty$

$$\epsilon_{eff} = \frac{\epsilon_h}{(1-f)^3}$$

There is experimental evidence that this result is fairly accurate at least for $0 \leq f \leq 0.5$.

3. Above a uniform earth, there is a horizontal electric dipole whose current density is $\mathbf{J}(\mathbf{r}, t) = IL\delta(x)\delta(y)\delta(z-h)e^{-i\omega t}\mathbf{e}_x$. The electromagnetic parameters of the air are $\mu_0, \epsilon_0, \sigma_0$ and in the earth $\mu_1, \epsilon_1, \sigma_1$. The earth's surface is the xy-plane.

   a) Show that the vector potential can be expressed as

$$
\begin{aligned}
A_{x0} &= C\int_0^\infty db\, \frac{b}{K_0}(e^{-K_0|z-h|} + R(b)e^{-K_0(z+h)})J_0(b\rho),\ z > 0 \\
A_{z0} &= C\frac{\partial}{\partial x}\int_0^\infty db\ S(b)e^{-K_0(z+h)}J_0(b\rho),\ z > 0 \\
A_{x1} &= C\int_0^\infty db\ P(b)e^{K_1 z}J_0(b\rho),\ z < 0 \\
A_{z1} &= C\frac{\partial}{\partial x}\int_0^\infty db\ Q(b)e^{K_1}J_0(b\rho),\ z < 0
\end{aligned}
$$

   where C is a constant to be determined later and

$$K = \sqrt{b^2 - k^2},\ \rho = \sqrt{x^2 + y^2}$$

   b) Express $\mathbf{B}$ and $\mathbf{E}$ using only the vector potential.

   c) Apply boundary conditions to solve P, Q, R, S.

   d) Construct a line current of amplitude I of successive dipoles. Calculate A_{x1} and A_{z1} (set $\mu_1 = \mu_o$).

   e) Calculate $\mathbf{B}_1$ for a line current. Determine the constant C by studying a time-independent current.

Reminder: there is still time to consider the extra problem no. 5 of exercise 3.