Applications of electrodynamics, spring 2005

Exercise 5 (Thursday 24.2., return answers until 16:00 on Monday 21.2.)

1. An infinitely long time-harmonic ($e^{-i\omega t}$) sheet current flows to the y direction on the xy-plane. The current density transverse to the flow direction is $f(x)$. The whole space has constant electromagnetic material parameters.

   a) Starting from the wave equation of the vector potential, show that the fields are

$$E_y = \frac{\omega\mu_0 I(\omega)}{4}\int_{-\infty}^{\infty} dx'\ f(x')H_0^{(2)}(-k\sqrt{(x'-x)^2+z^2})$$

$$B_x = \frac{i\mu_0 k I(\omega)}{4}\int_{-\infty}^{\infty} dx'\ f(x')\frac{zH_1^{(2)}(-k\sqrt{(x'-x)^2+z^2})}{\sqrt{(x'-x)^2+z^2}}$$

$$B_z = \frac{i\mu_0 k I(\omega)}{4}\int_{-\infty}^{\infty} dx'\ f(x')\frac{(x'-x)H_1^{(2)}(-k\sqrt{(x'-x)^2+z^2})}{\sqrt{(x'-x)^2+z^2}}$$

   b) Show by a direct calculation that the Maxwell equations are fulfilled.
   c) Show that at the static limit the expressions of the magnetic field reduce to the Biot and Savart law.

2. A time-harmonic line current flows above a uniform earth at the height h. Assuming low frequencies, use the complex image method to study the behaviour of the surface fields and the ratio E_{hor}/B_{hor} at large distances from the current.

3. A line current I flows above a layered earth at height $z = -h$. The earth's surface is the xy-plane and z is positive downwards.
a) Using the integrals 5.38-5.40 and omitting the displacement current, show that the surface fields can be written as

$$B_x(x,\omega) = \frac{\mu_0 I}{2\pi}\int_0^\infty (e^{-bh} + Re^{-bh})\cos bx \; db$$

$$B_z(x,\omega) = -\frac{\mu_0 I}{2\pi}\int_0^\infty (e^{-bh} - Re^{-bh})\sin bx \; db$$

$$E_y(x,\omega) = \frac{i\omega\mu_0 I}{2\pi}\int_0^\infty (1-R)\frac{e^{-bh}}{b}\cos bx \; db$$

where the reflection coefficient is

$$R = \frac{i\omega\mu_0/Z + b}{i\omega\mu_0/Z - b}$$

b) Show that the complex image depth is

$$p = -\frac{Z}{i\omega\mu_0}$$

assuming that $|pb|^3 \ll 1$. Tip: apply the binomial expansion to

$$R = 1 - \frac{2pb}{1+pb}$$

and compare to the power series of e^{-2pb}.