Introduction to inverse problems

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Outline

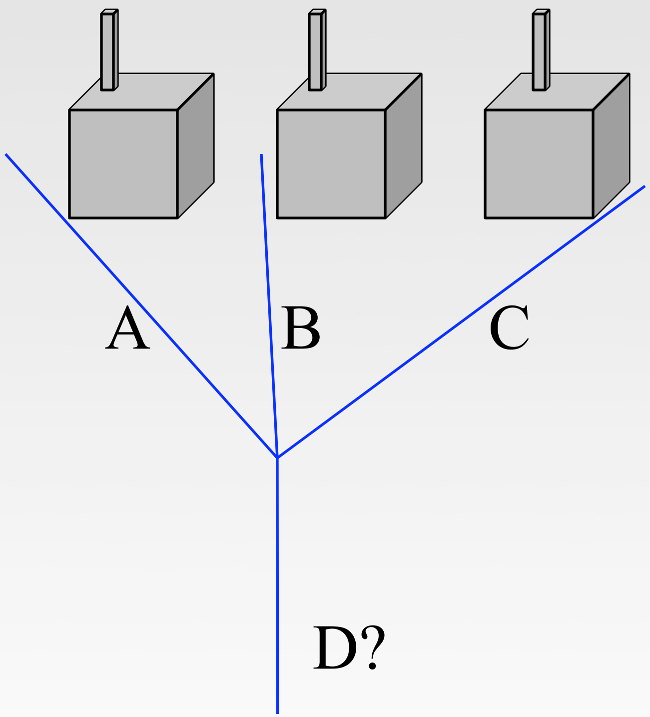
- What are inversion problems?
- Where are they used?
- And how are they solved?
- One example in detal: Doppler imaging

What is an inverse problem?

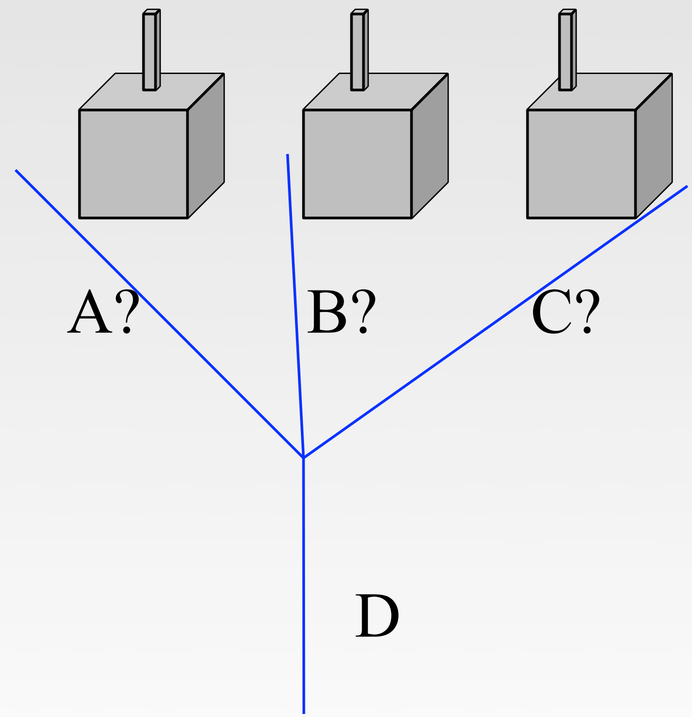
Alifanov: Solution of an inverse problem entails determining unknown *causes* based on observation of their *effects*.

Or: A problem where the answer is known, but not the question

Forward vs. Inverse problem



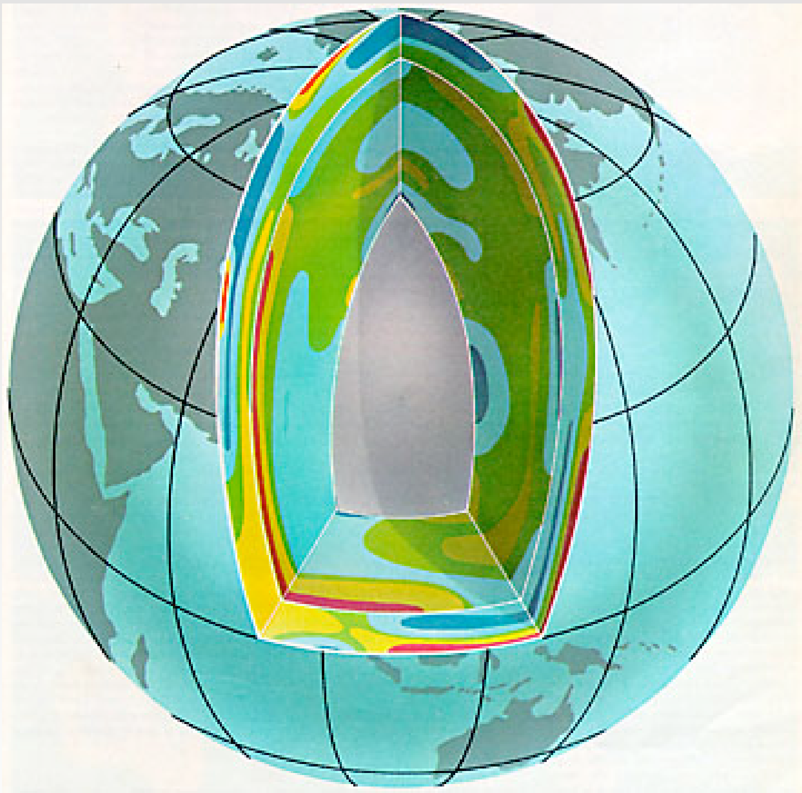
Forward problem



Inverse Problem

Applications of inverse methods

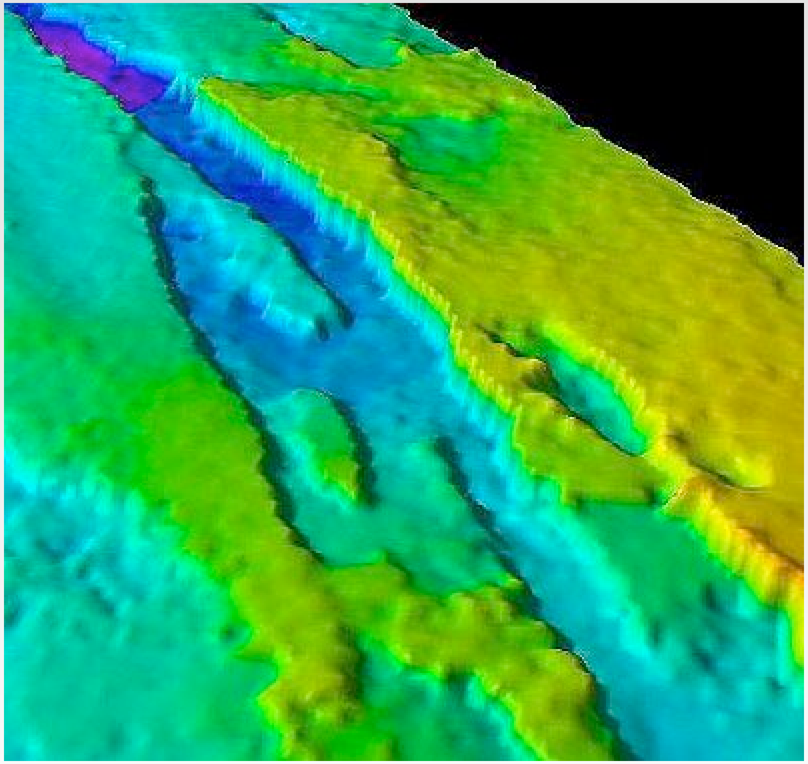
Seismology



- The scientific study of earthquakes and the propagation of elastic waves through the Earth
- Also includes studies of earthquake effects
- Earthquakes, and other sources, produce different types of seismic waves that travel through rock, and provide an effective way to image both sources and structures deep within the Earth
- ‚Related‘ subjects: helio- and asteroseisomolgy

Seismic surveys

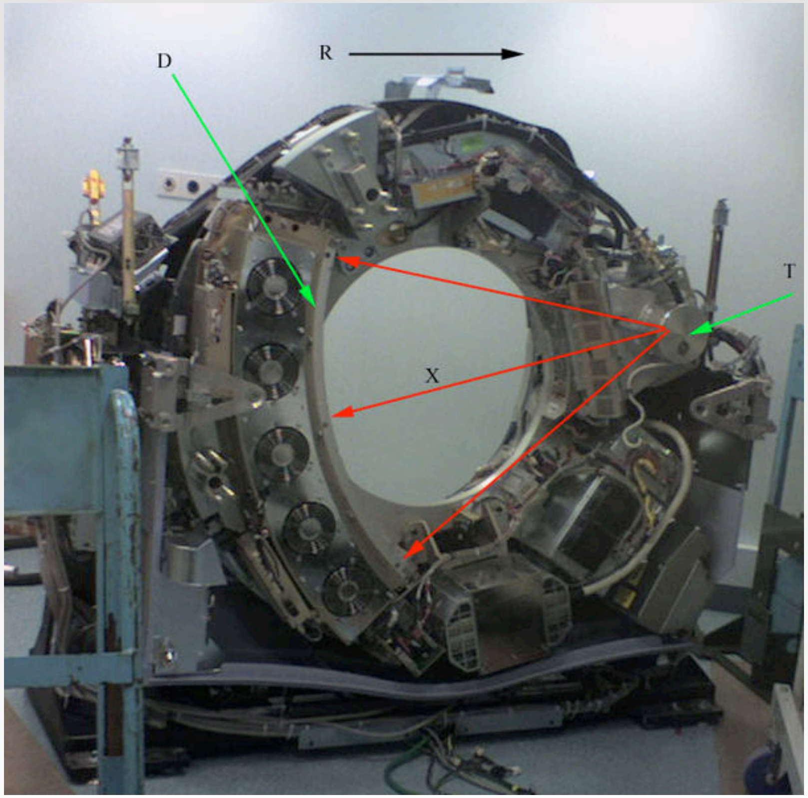
- For locating ground water
- Investigating locations for landfills
- Characterizing how an area will shake during an earthquake
- For oil and gas exploration.



3D model of the top reservoir structure of an oilfield (Leeds University)

Computerized Tomography (CT Scan)

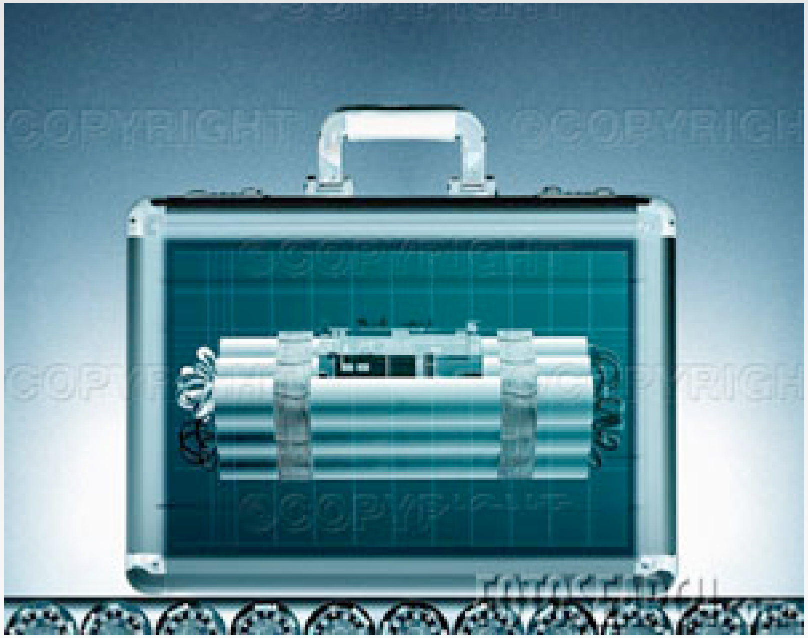
- Method employing tomography
- Dgital geometry processing is used to generate a three-dimensional image of the internals of an object
- From a large series of two-dimensional X-ray images taken around a single axis of rotation.



CT scanner

Other applications

- Airport security
- Industrial process monitoring
- Photoelasticity - visualisation of the stress inside a transparent object
- Electromagnetic monitoring of molten metal flow
- X-ray tomography in material science
- Numerous applications in physics and astronomy
- etc



From Fotosearch©

Solving inverse problems

Three fundamental questions

- How accurately are the data known?
- How accurately can we model the response of the system?
- What is known about the system independent of the data?

A priori information

- Often very important
- For any sufficiently fine parameterization of a system there will always be unreasonable models that fit the data too
- Prior information is the means by which the unreasonable models are rejected or down-weighted

General formulation

- Inverse problem can be formulated:

$$d=G(m)$$

- where *G* is an operator describing the explicit relationship between data (d) and model parameters (m)
- G is a representation of the physical system
- For linear problems d and m are vectors and G a matrix

Well-posed problem

- Stems from a definition given by Hadamard
  - A solution exists
  - The solution is unique
  - The solution depends continuously on the data, in some reasonable topology
- Problems that are not well-posed in the sense of Hadamard are termed **ill-posed**
- Inverse problems are often ill-posed.

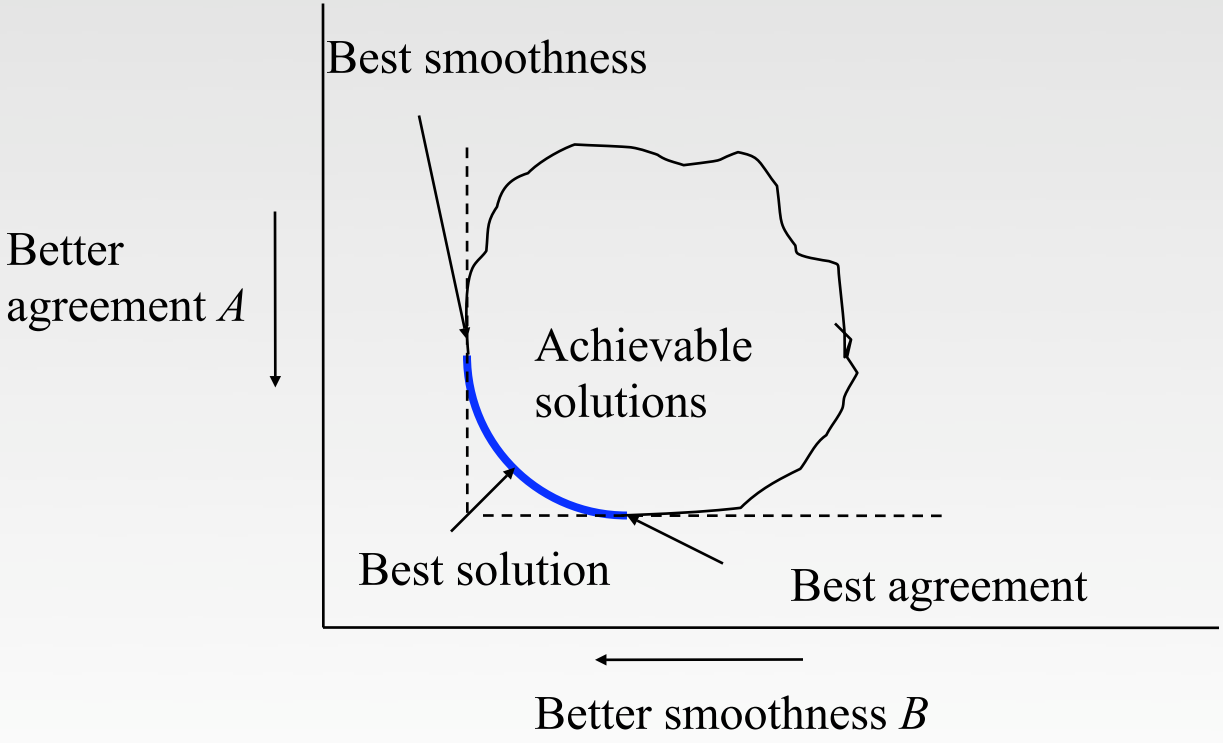
Regularisation

- If a problem is well-posed, then it stands a good chance of solution on a computer using a stable algorithm
- If it is not well-posed, it needs to be re-formulated for numerical treatment
- Typically this involves including additional assumptions, such as smoothness of solution
- This process is known as **regularisation**

Regularisation II

- In general ill-posed inverse problem can be thought to consist of two positive functionals *A* and *B*
- *A* measures agreement of the model to the data. If *A* alone is minimised the agreement becomes impossibly good, but the solution is unstable or unrealistic
- *B* measures ‚smoothness' of the desired solution. Minimising *B* gives a solution that is ‚smooth' or ‚stable' or ‚likely' – and has nothing to do with the measurements

Agreement vs. smoothness



Maximum Entropy Principle

$$r(X(M))=\iint_M X(M)\log(X(M))dM$$

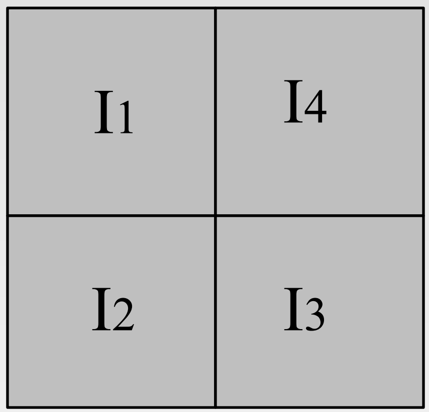
- Provides uniqueness of the solution and minimum correlation between the elements
- This way we get the X(M) with the highest possible informational entropy

Tikhonov regularisation

$$r_T(X)=\iint_M |\text{ grad } X(M)\,|\text{\^{}}2\ dM$$

- With this regularisation we are looking for the smoothest possible solution that still produces the observations
- Used in the cases where strong correlation between neighbouring points is expected

Student exercise



Observations:

$I_1+I_2+I_3+I_4=36$

$I_1+I_2=12$

$I_1+I_4=12$

Calculate I_1, I_2, I_3 and I_4 using both Tikhonov regularisation and Maximum Entropy Principle

Some hints

Tikhonov: r_T=(I1-I2)^2+(I2-I3)^2+(I3-I4)^2+(I4-I1)^2

MEP: r_{MEP}=I1log(I1)+I2log(I2)+I3log(I3)+I4log(I4)

Express I2, I3 and I4 in respect to I1 and then differentitate

Results

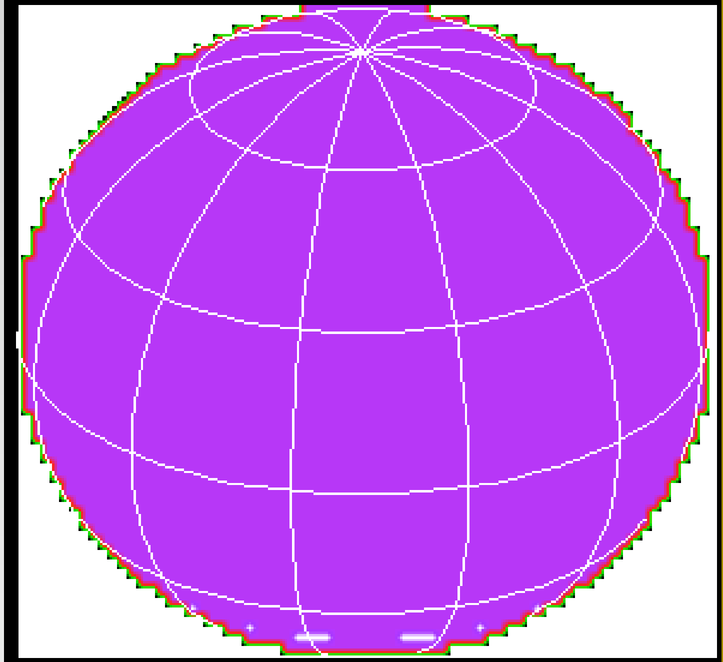
Tikhonov: I1=3, I2=I4=9, I3=15

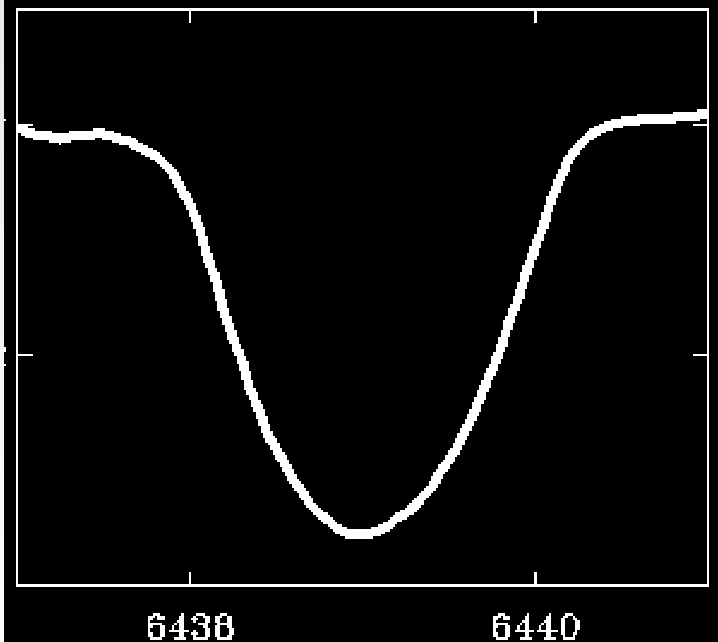
MEP: I1=4, I2=I4=8, I3=16

- Tikhonov regularisation produces plane brightness distribution (I3-I2=I2-I1)
- Maximum Entropy Principle produces a bright spot in the lower corner (I3=I2+I4)

Doppler imaging

- Used for making spatially resolved maps of the stellar surface
- Mapped characteristic can for example be:
  - Effective temperature
  - Elemental abundance
  - Magnetic field
- Invented by Deutsch (1958) and developed further by Deutsch (1970), Falk \& Wehlau (1974), Goncharsky et al. (1977) and many others





From Berdyugina

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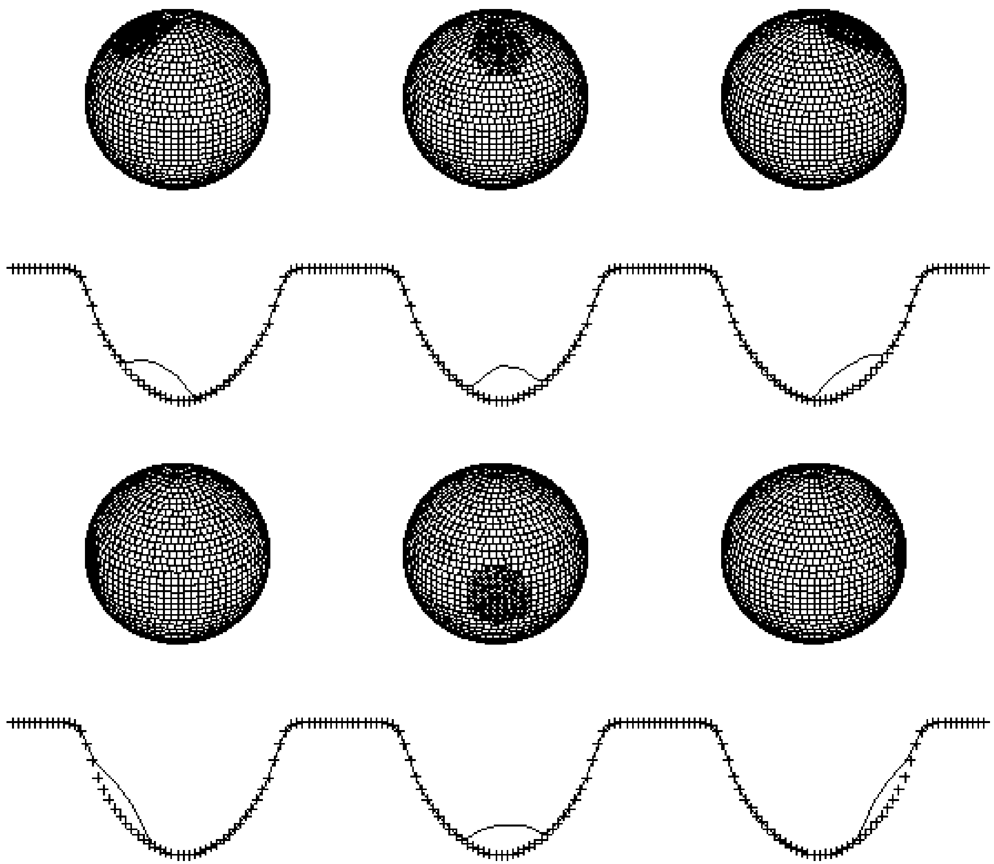

From Hatzes

Formulation of the problem I

- X(M) characterises the strength of the spectral line originating from the surface point M
- X(M) can be temperature, abundance, etc
- The residual lineprofile at some phase (θ) can be calculated by:

$$R_{obs}(\lambda,\varphi) = \frac{\int\int I_c(\lambda, X(M), M) R_{loc}(\lambda - \Delta\lambda(M), X(M), M) dS(M)}{\int\int I_c(\lambda, X(M), M) dS(M)}$$

$R_{loc} = 1 - I_{line}/I_{cont}$

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Formulation of the problem II

- The inverse problem amounts to finding the surface parameter X(M) from the observed profiles such that

$$D_\lambda(\varphi) = \frac{1}{n_\varphi n_\lambda} \sum_{\varphi=1}^{n_\varphi} \sum_{\lambda=1}^{n_\lambda} g(\varphi,\lambda)[R_{obs}(\varphi,\lambda) - R_{th}(\varphi,\lambda)]^2 \le \sigma^2$$

- As this is an ill-posed problem we need regularisation

$$F(X) = D + \Lambda r(X(M)) = \text{minimum}$$

AIP

Requirements

- Models
  - Accurate line profile modelling
- Instrumentation
  - High spectral resolution
  - High signal-to-noise ratio
- Object
  - Good phase coverage (convenient rotation period)
  - Rapid rotation
  - Not too long exposure time (bright)
  - Something to map!

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AIP

Resolution

- The best resolution on the stellar surface is achieved when:

$$FWHM_{instr} \leq FWHM_{line}$$

- The intrinsic line profile is significantly broadened even if the star would not rotate. For a solar-type star with T_{eff}=5000 K the thermal line width is 1.2 km/s
- Instrumental resolution $\lambda/\Delta\lambda \approx 35000$-120000 corresponds to 8-3 km/s

Signal-to-noise of the observations

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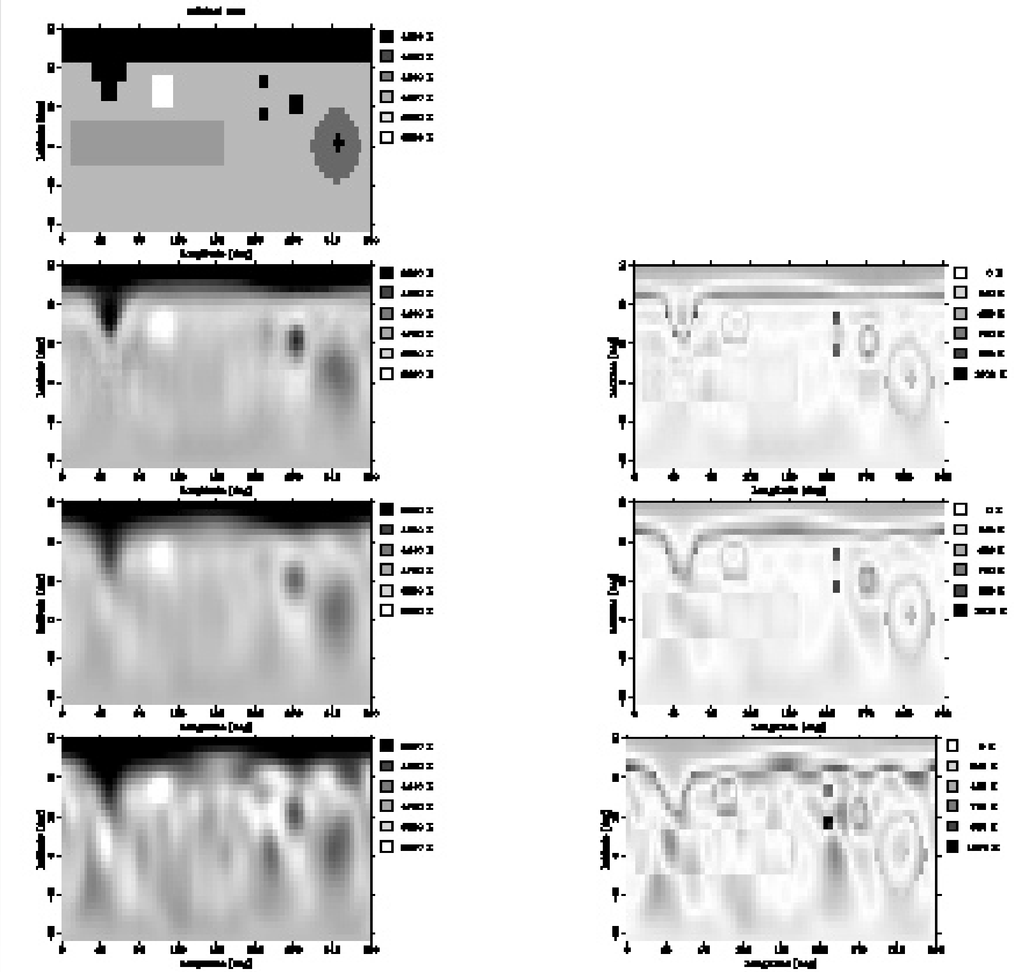
S/N = 3000

S/N =
300

S/N = 75



Strassmeier 2001

Phase coverage

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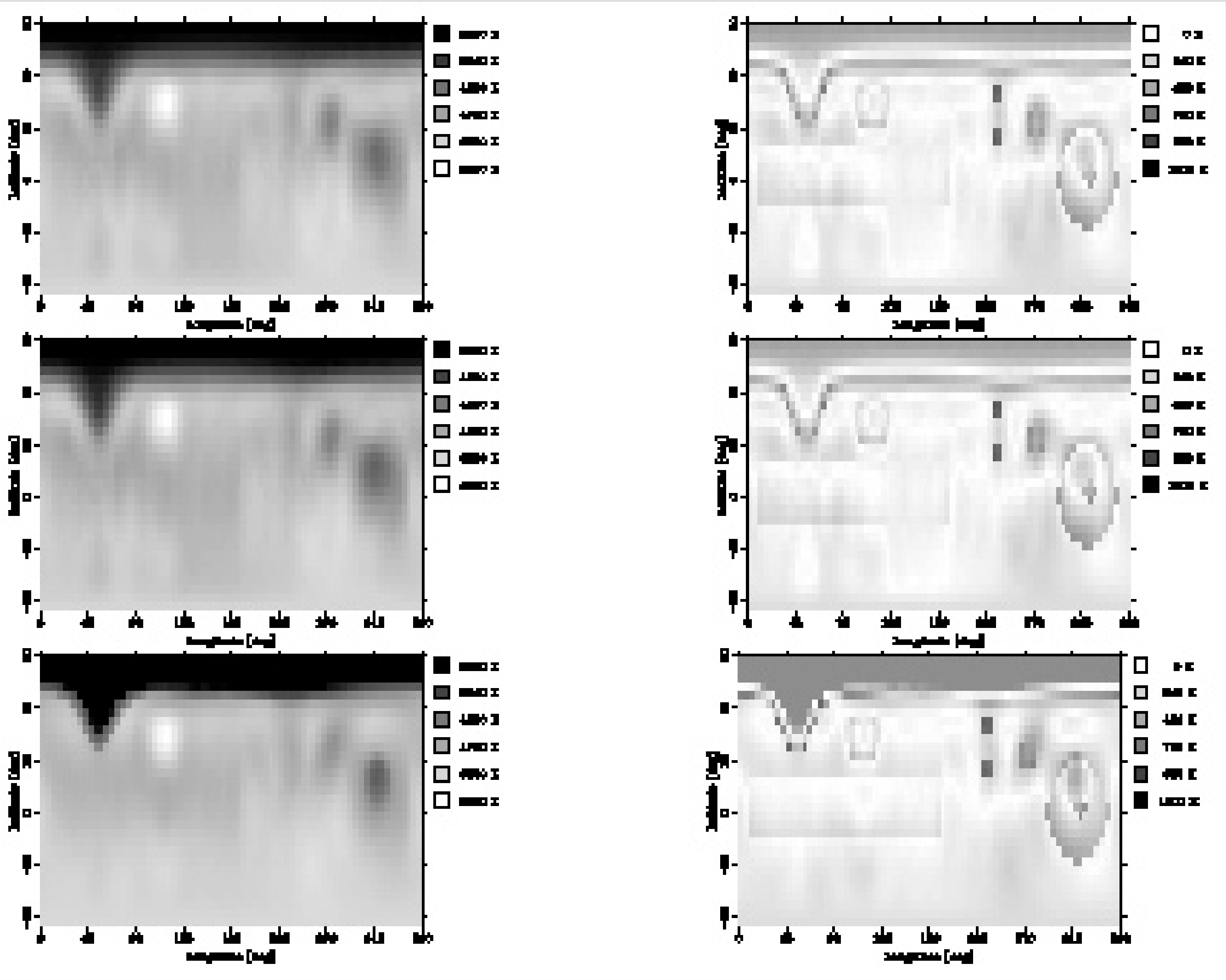
GAP

lon = 305°-5°

lon=285°-5°

lon=265°-5°



Strassmeier 2001

Simulations by Silva Järvinen

OBJECT

vsin
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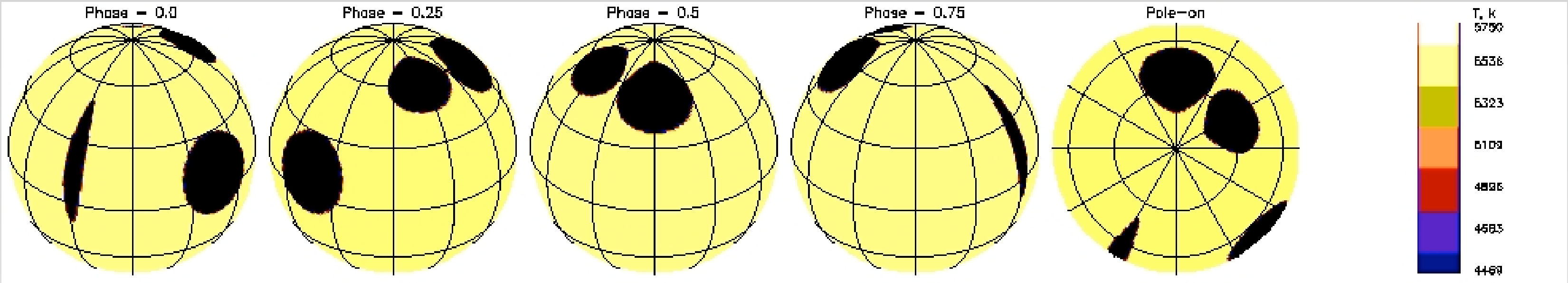
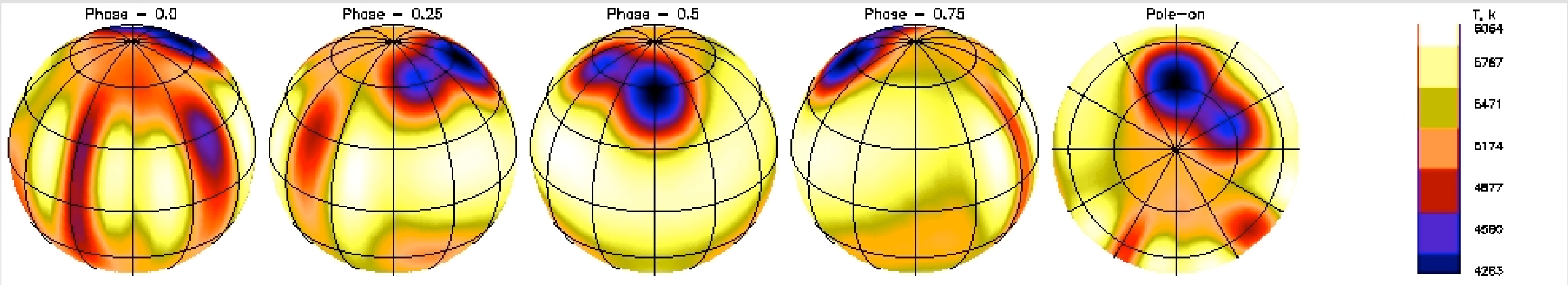
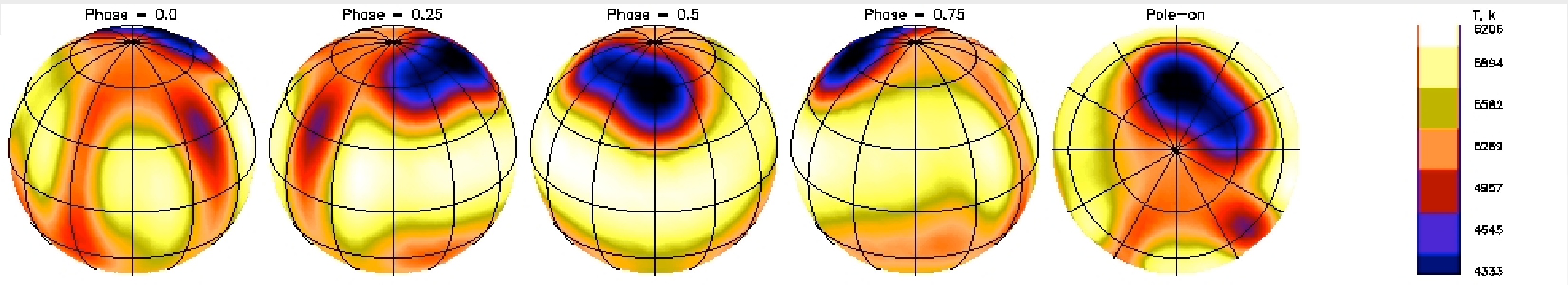
Resolution: 250 000
S/N: infinite
Phases: 20



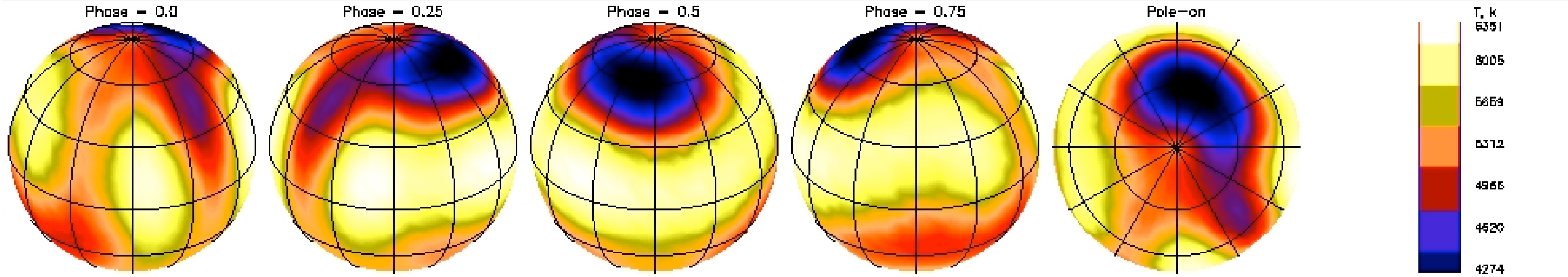
Vsini = 45 km/s



Vsini = 30 km/s



Vsini = 17 km/s



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Phase smearing

- During the observations the star rotates and the bump moves in the lineprofile

→ The bump signal will be smeared

- Integration time should be as short as possible:

$$\Delta t \leq 0.01 P_{rot}$$

Examples:

$P_{rot} = 20$ days	$\Delta t > 4.5$ hours
$P_{rot} = 5$ days	$\Delta t \approx 70$ minutes
$P_{rot} = 2$ days	$\Delta t \approx 30$ minutes
$P_{rot} = 0.5$ days	$\Delta t \approx 7$ minutes

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Spot size

- Largest observed sunspot groups extend about 5° degrees (radius)
- What effects would they have in the lineprofiles?

Simulations

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10º spot (radius)

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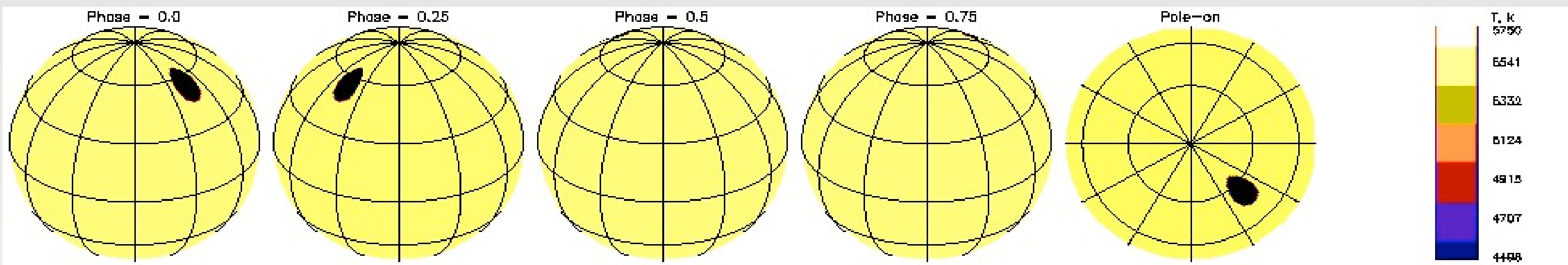
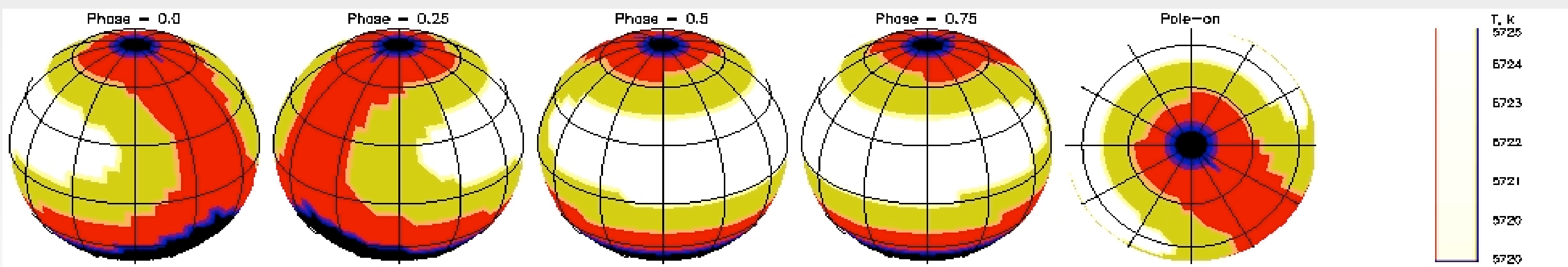
With Doppler imaging:

Resolution: 250 000
S/N: infinite
Phases: 20





Simulations by Silva Järvinen

Doppler imaging cannot be used for studying solar type spot groups

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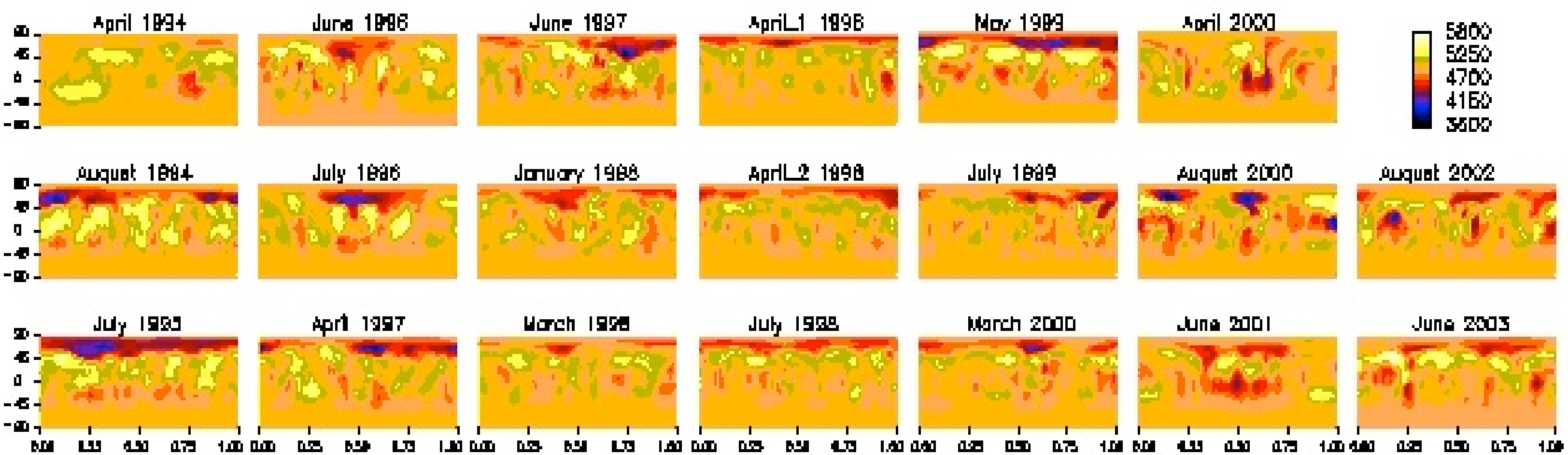
Temperature maps of FK Com for 1994-2003



Ending note

Inversion methods are powerful tools BUT

- Take great care when obtaining observations
- Take time to understand your observations and their limitations
- Your model is crucial, so think carefully that you have included all the necessary physics

When all these points are taken into account you can produce very interesting science