

The Finnish Graduate School in Astronomy and Space Physics Summer School 2007:

Time Series Analysis

Part III. Methods

Numerical and statistical methods in
time series analysis

Why is there so many peaks in spectra?

- Fourier transform is complex transform
- Data set is finite
- We can compute only approximately
- Data set is not equally spaced
- Signal is not harmonic
- The signal can contain more frequencies
- Signal parameters change in time
- Observations are noisy

Fourier transform is complex transform

Greatest Equations

Physics World magazine recently asked readers to send in nominations for the best equations of all time. Euler's equation was one of the most popular. It has wide application in understanding the motion of any type of wave, including light.

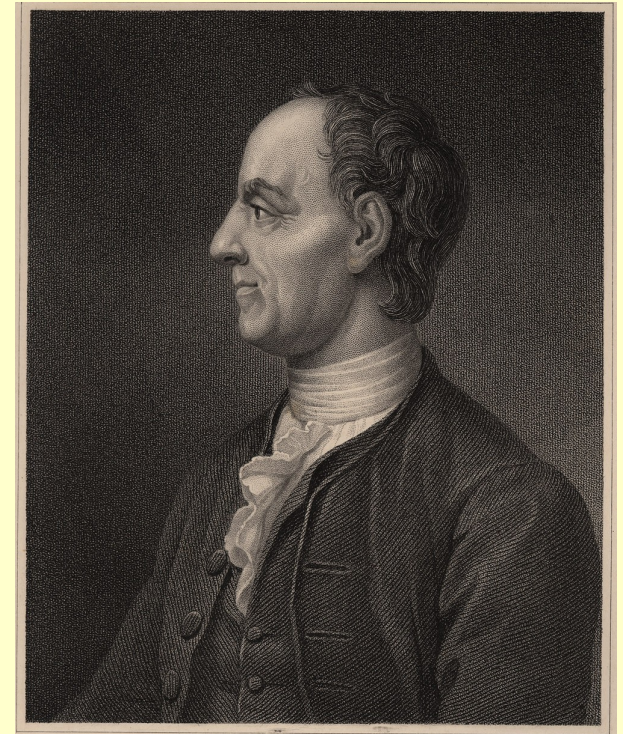
Euler's equation *Contains nine basic concepts of mathematics, elegantly.*

Diagram illustrating Euler's equation: $e^{i\pi} + 1 = 0$. The components are labeled as follows:

- EXONENTS**: Points to the $i\pi$ term.
- THE SQUARE ROOT OF -1 Imaginary**: Points to the i .
- PI=3.14159...**: Points to the π .
- BASE OF NATURAL LOGARITHMS =2.71828...**: Points to the e .
- MULTIPLICATION**: Points to the $i\pi$ term.
- ADDITION ONE**: Points to the $+ 1$.
- EQUALS**: Points to the $=$.
- ZERO**: Points to the 0 .

One respondent said of this equation:

"What could be more mystical than an imaginary number interacting with real numbers to produce nothing?"



Leonhard Paul Euler

April 15, 1707

September 18 [O.S. September 7] 1783

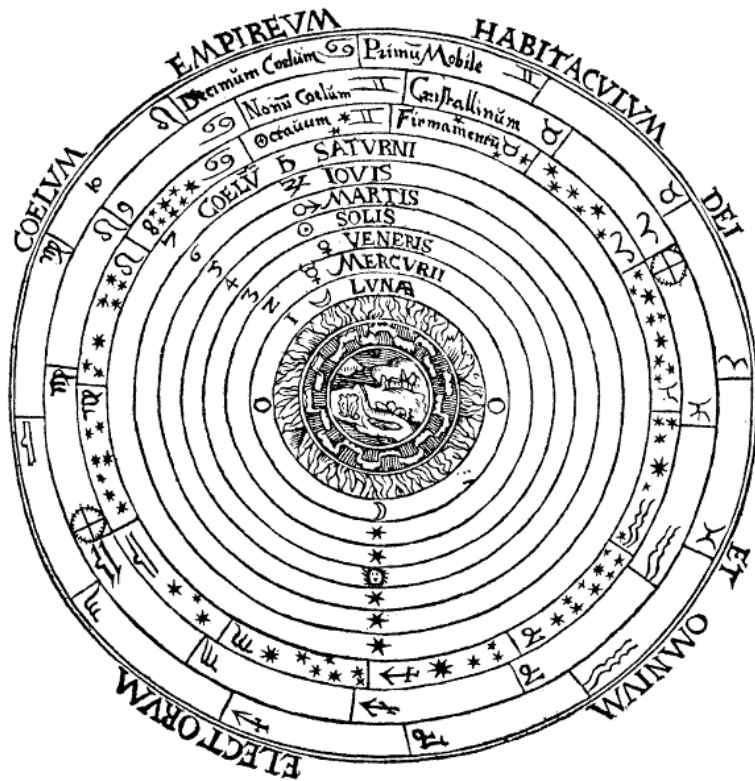
But harmonics are real!



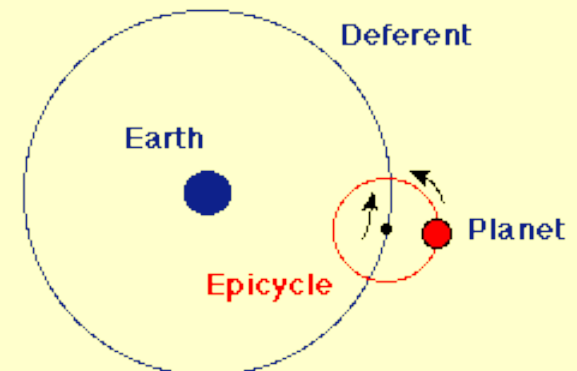
Claudius Ptolemaeus (Ptolemy)
ca. 90 – ca. 168 AD

Abstract clock = $C(\nu)$

Schema huius præmissæ diuisionis Sphærarum .



- Clock
- Cisoid (cosinusoid + sinusoid)
- Complex harmonic
- Circle (of unit radius)
- Circular harmonic
- Cycle
- etc.



The most important algebraic property of abstract clocks

$$C(\nu_1)C(\nu_2) = C(\nu_1 + \nu_2)$$

$$C(\nu)C(-\nu) = C(0)$$

Just clock which is not working (rotating) at all

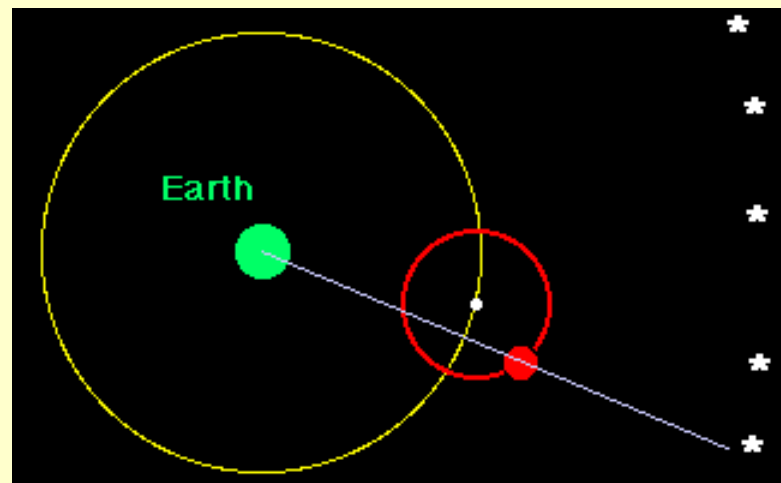
Phases and rotators

$$\text{Phase} = p(\phi)$$

$$p(\phi_1)p(\phi_2) = p(\phi_1 + \phi_2)$$

$$\text{Rotator} = Ap(\phi)C(\nu)$$

$$\text{Orbit} = \sum_{\nu} A_{\nu} p(\phi_{\nu}) C(\nu) + \sum_{\nu} A_{\nu} p(\phi_{\nu}) C(\nu)$$



Definition of trigonometric functions

$$\cos(\nu) = \frac{C(\nu) + C(-\nu)}{2}$$

Similarly

$$\frac{C(\nu) - C(-\nu)}{2}$$

Using “back-rotation”

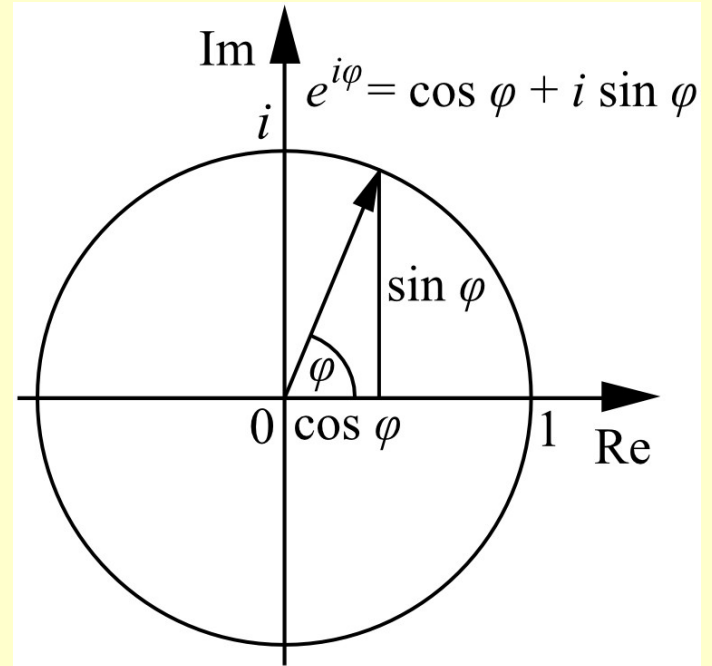
$$j^4 = p(0)$$

$$\sin(\nu) = \frac{C(\nu) - C(-\nu)}{2j}$$

Euler's formula again

$$e^{j2\pi\nu t} = \cos(2\pi\nu t) + j \sin(2\pi\nu t)$$

- Natural number 2.
- Real number ν .
- Two transcendental numbers e and π .
- Imaginary number j .
- Exponentiation.
- Two trigonometric functions.



The Fourier Transform and its Inverse in another format

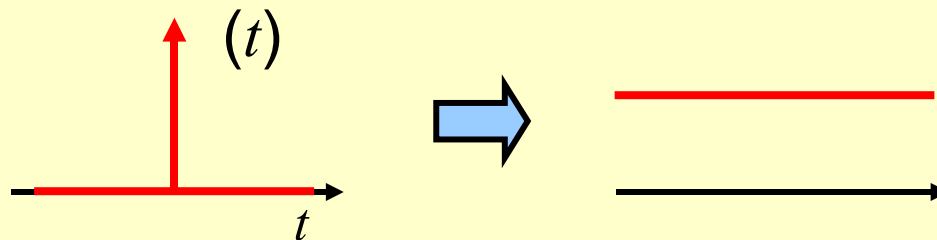
$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) d\omega$$

The Fourier Transform of $\delta(t)$ is 1

$$\int_{-\infty}^{\infty} \delta(t) \exp(-i\omega t) dt = \exp(-i\omega[0]) = 1$$

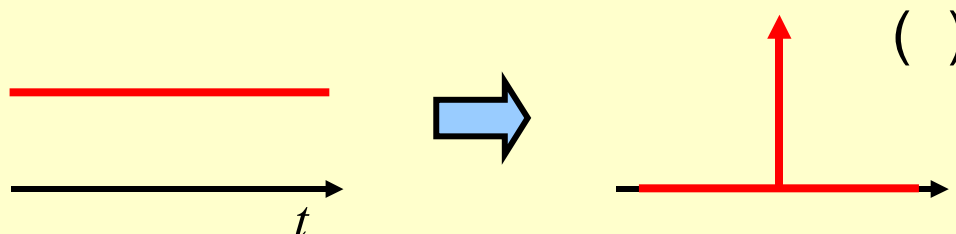
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And the Fourier Transform of 1 is

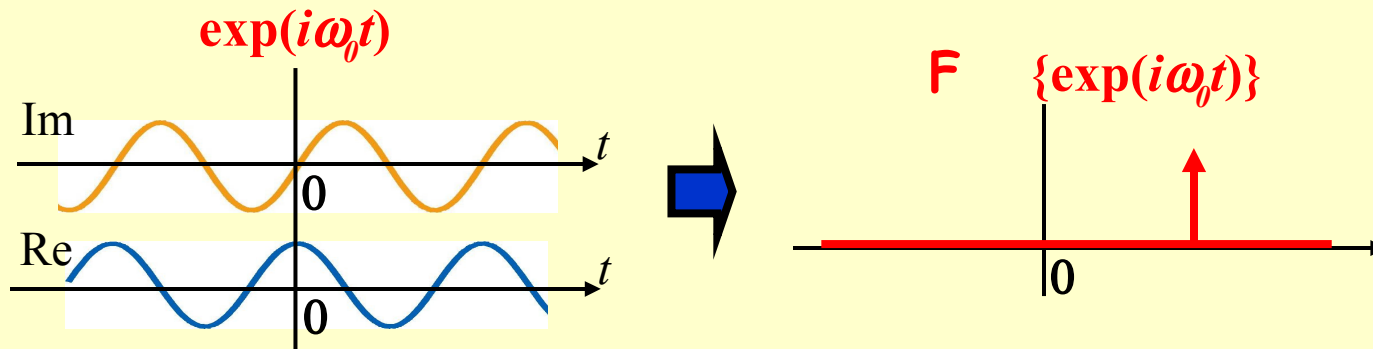
$$\int_{-\infty}^{\infty} 1 \exp(-i\omega t) dt = 2\pi \delta(\omega)$$

-



The Fourier transform of $\exp(i\omega_0 t)$

$$\begin{aligned}
 \mathcal{F} \{ \exp(i\omega_0 t) \} &= \int_{-\infty}^{\infty} \exp(i\omega_0 t) \exp(-i\omega t) dt \\
 &= \int_{-\infty}^{\infty} \exp(-i[\omega - \omega_0]t) dt = 2\pi \delta(\omega - \omega_0)
 \end{aligned}$$



The function $\exp(i\omega_0 t)$ is the essential component of Fourier analysis.

It is a pure frequency.

The Fourier transform of $\cos(\omega_0 t)$

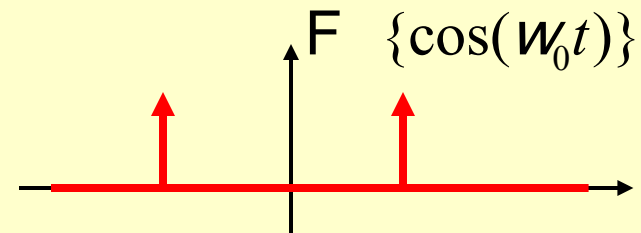
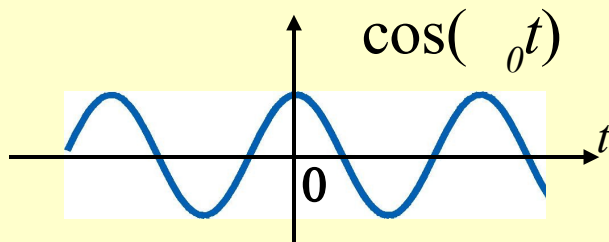


$$F \{ \cos(\omega_0 t) \} = \int_{-\infty}^{\infty} \cos(\omega_0 t) \exp(-i \omega t) dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} [\exp(i \omega_0 t) + \exp(-i \omega_0 t)] \exp(-i \omega t) dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \exp(-i[\omega - \omega_0]t) dt + \frac{1}{2} \int_{-\infty}^{\infty} \exp(-i[\omega + \omega_0]t) dt$$

$$= p \delta(\omega - \omega_0) + p \delta(\omega + \omega_0)$$



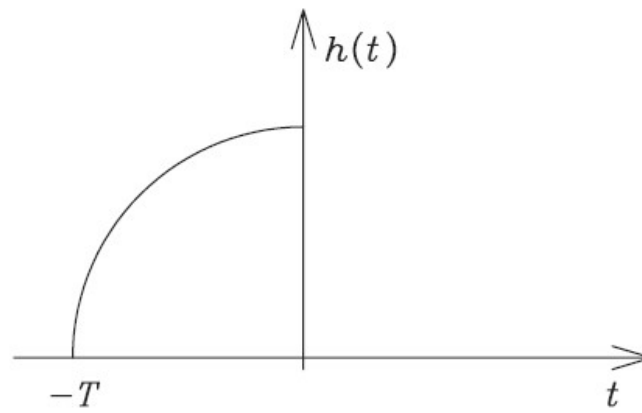
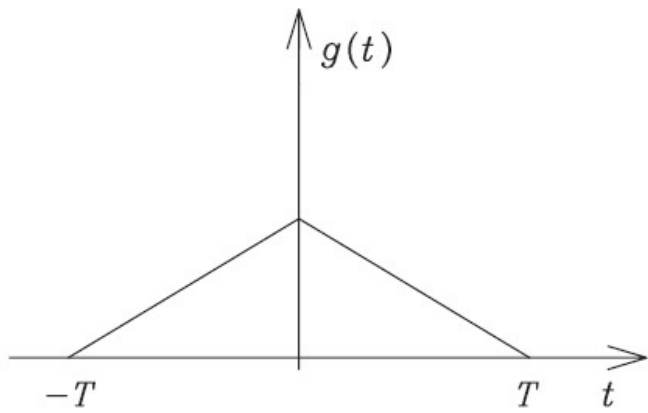
Convolution

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x - u)du$$

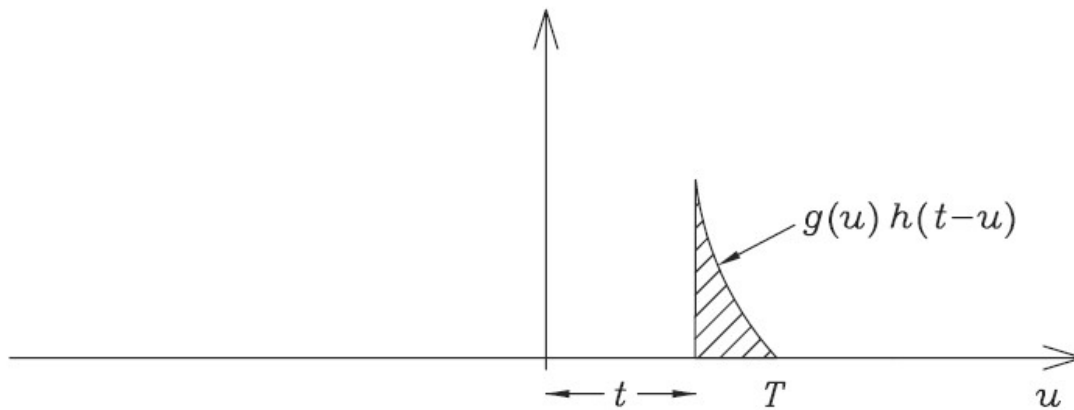
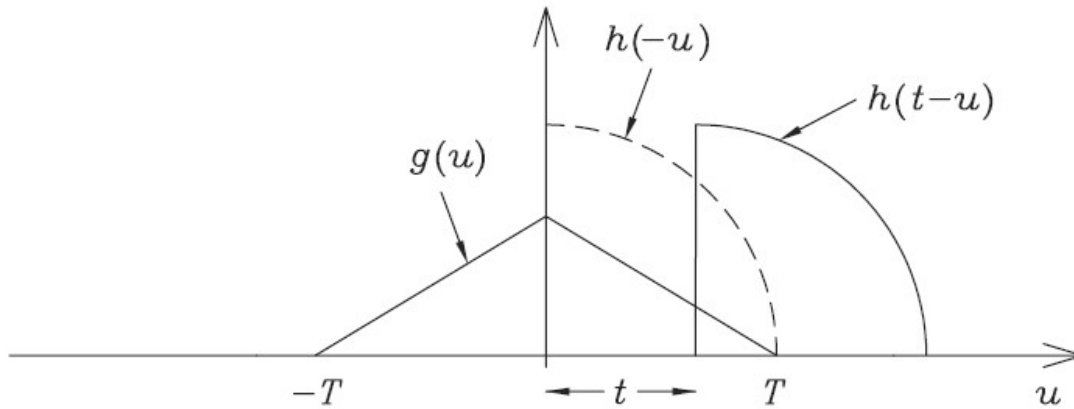
If

$$\mathcal{F}(f(t)) = F(\nu) \text{ and } \mathcal{F}(g(t)) = G(\nu)$$

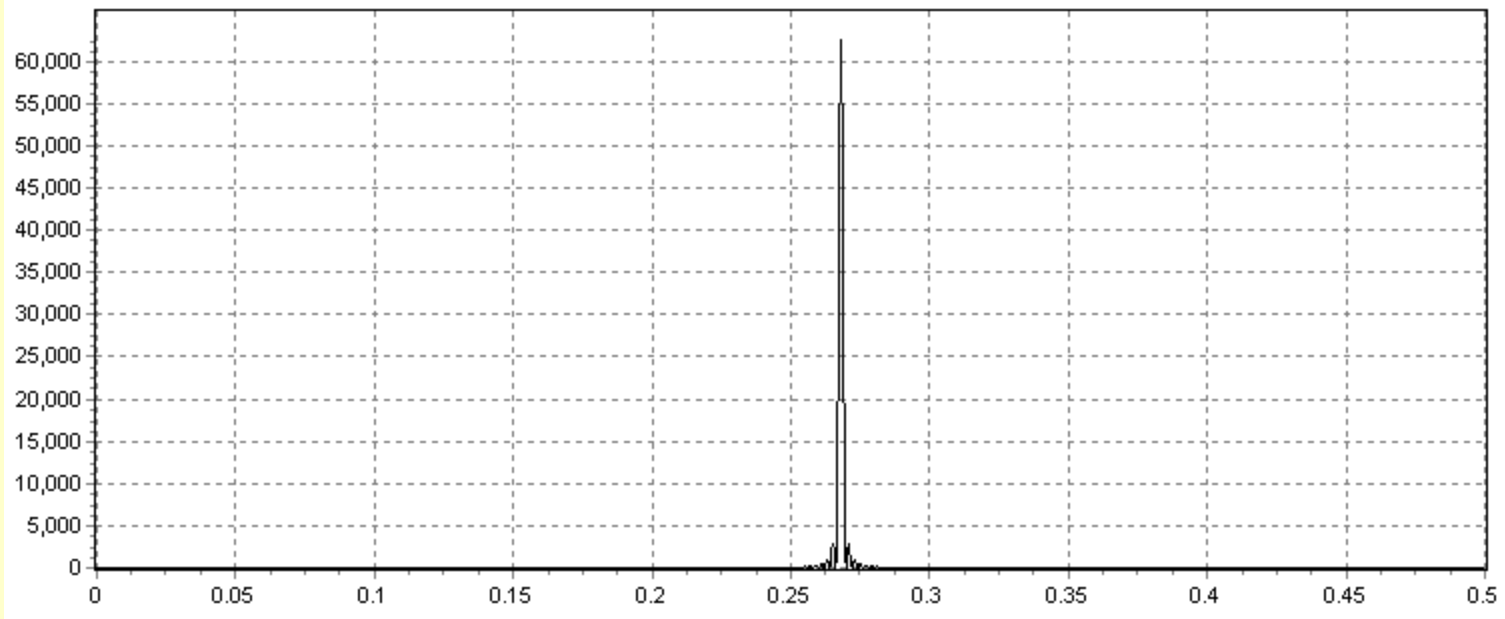
$$\mathcal{F}(f(t) * g(t)) = F(\nu)G(\nu)$$



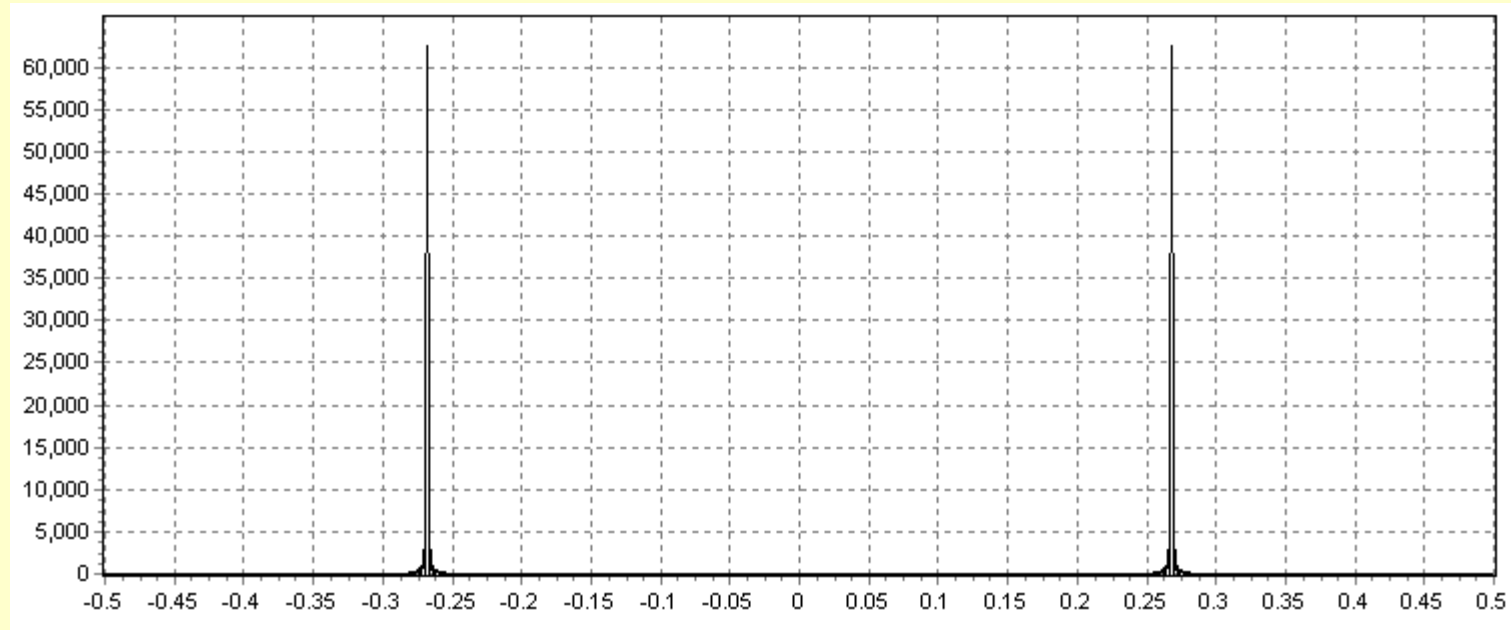
How
Convolution
works?



Ideal case, one component, one peak

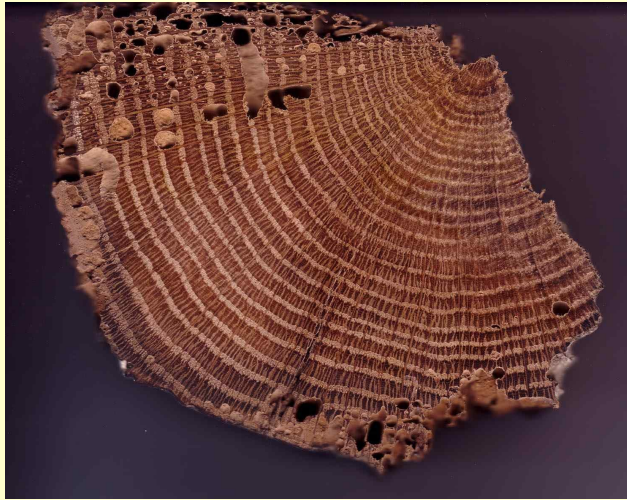


Fourier transform is complex, harmonics are real

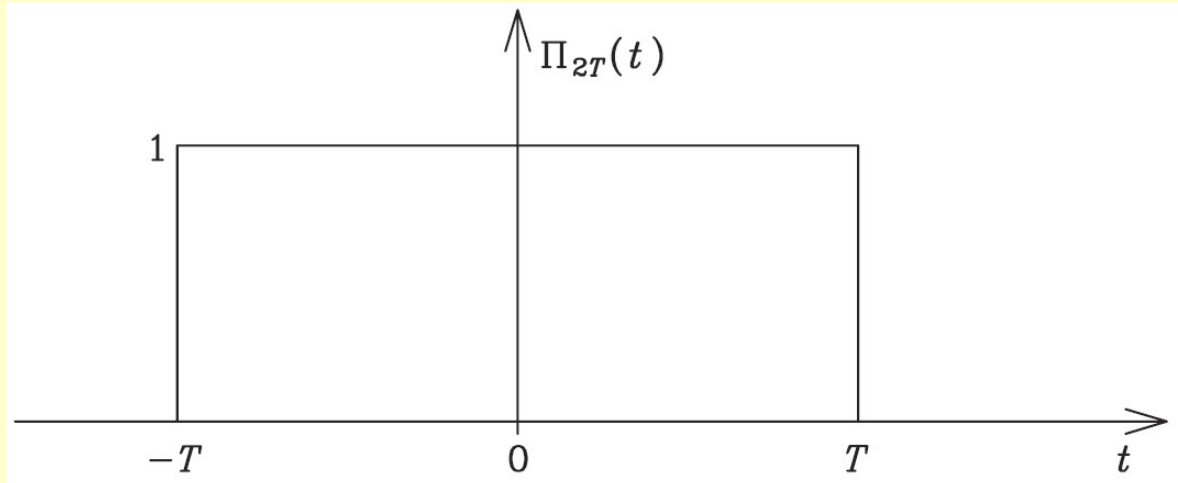


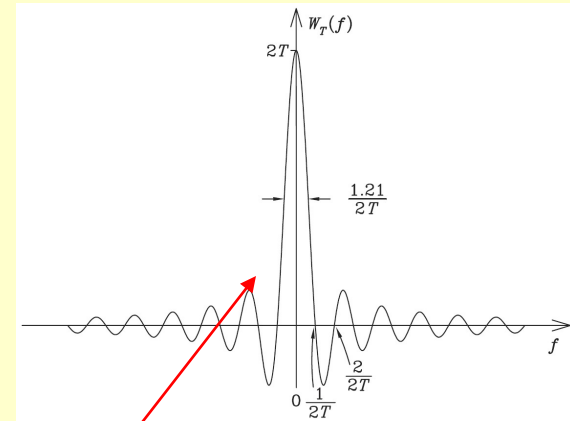
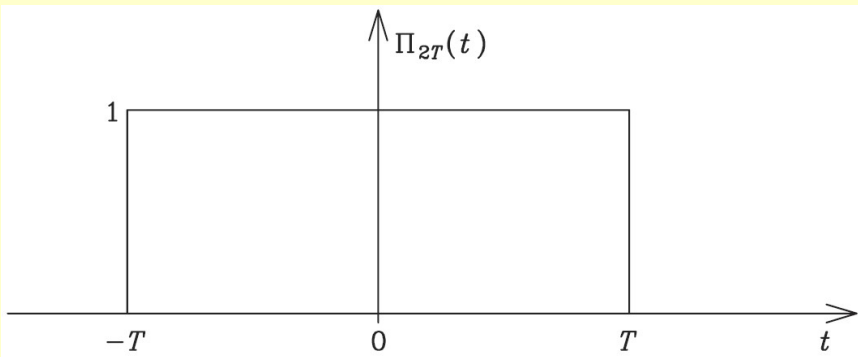
$$\cos(\nu) = \frac{C(\nu) + C(-\nu)}{2}$$

Data sets are finite



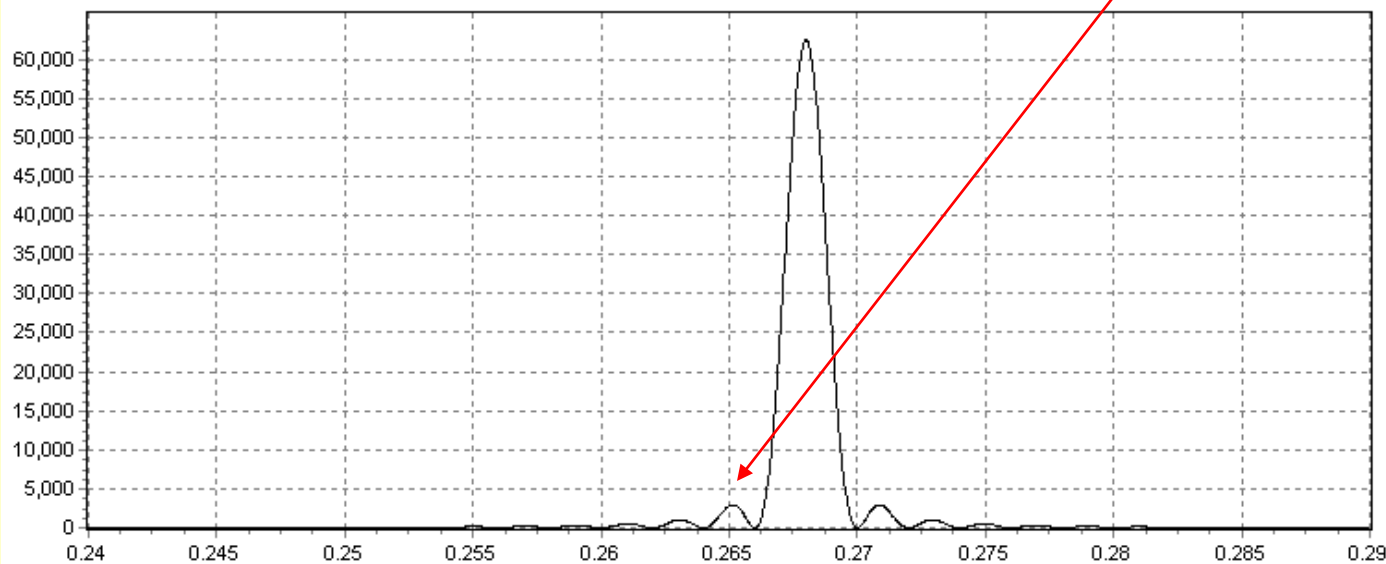
$$\Pi(x) = \left\{ \begin{array}{ll} 0, & |x| > \frac{1}{2} \\ \frac{1}{2}, & |x| = \frac{1}{2} \\ 1, & |x| < \frac{1}{2} \end{array} \right\}$$





$$\Pi(x) = \begin{cases} 0, & |x| > \frac{1}{2} \\ \frac{1}{2}, & |x| = \frac{1}{2} \\ 1, & |x| < \frac{1}{2} \end{cases}$$

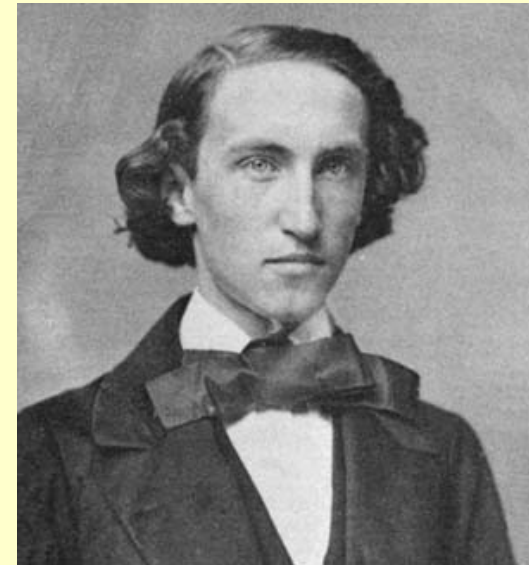
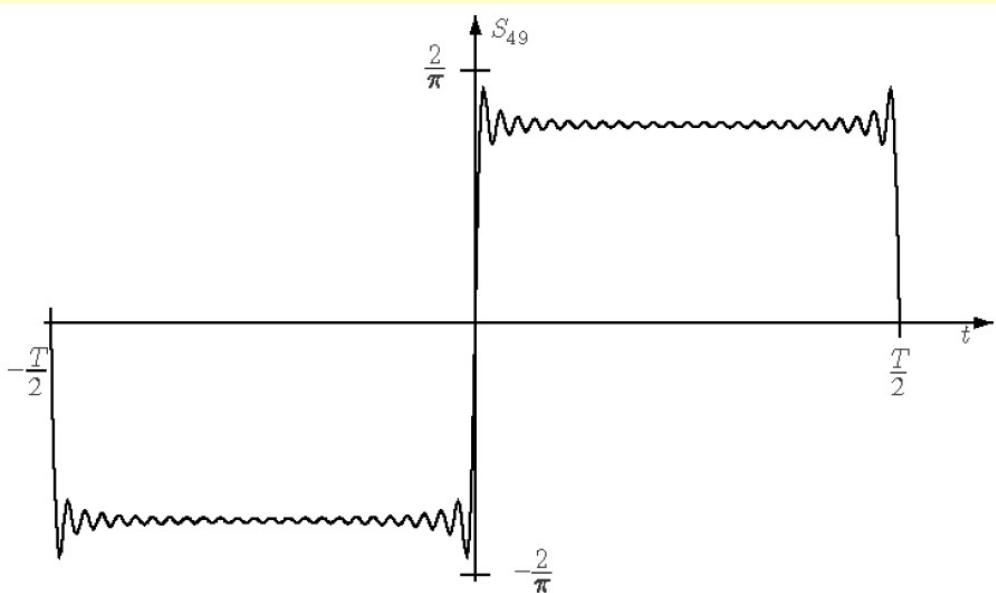
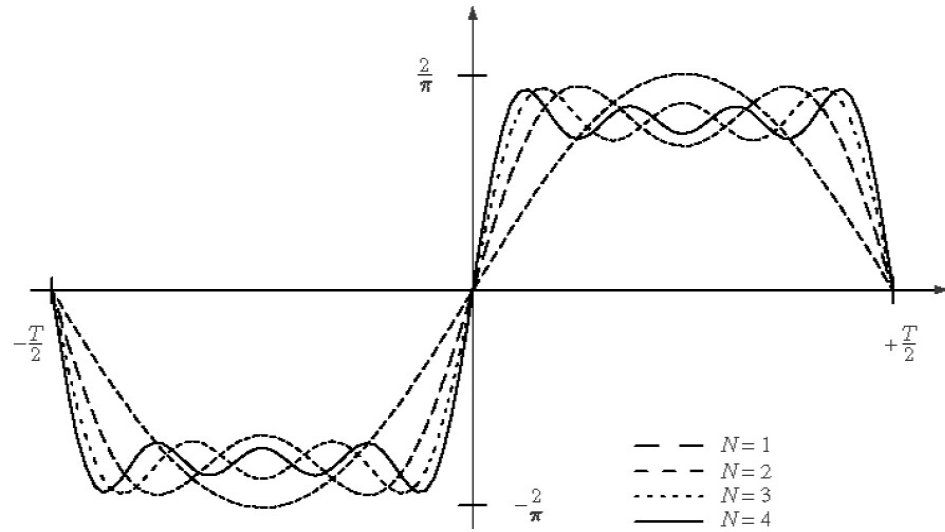
$$\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$$



More
peaks

We can compute only approximately

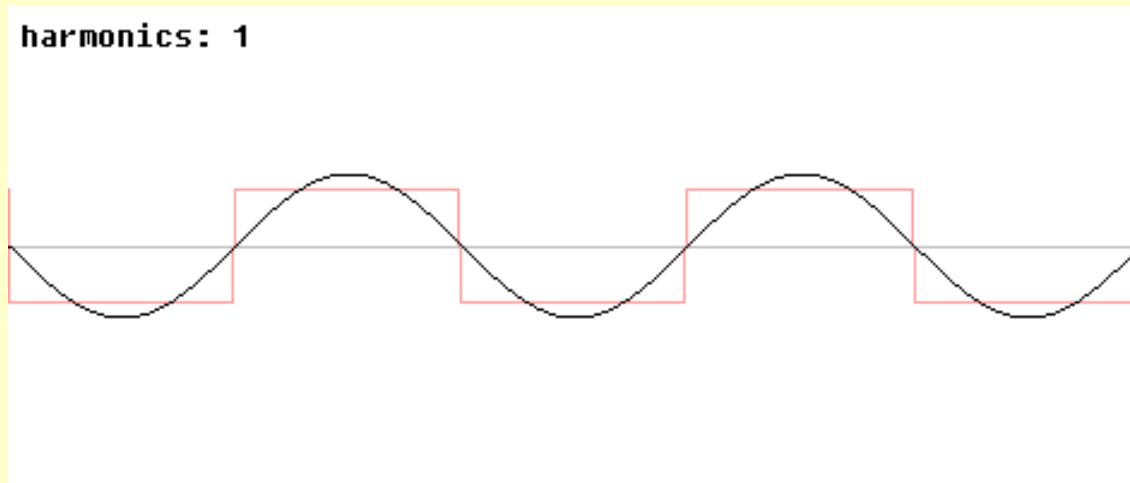
- Gibbs' phenomenon
- Gibbs' overshoot



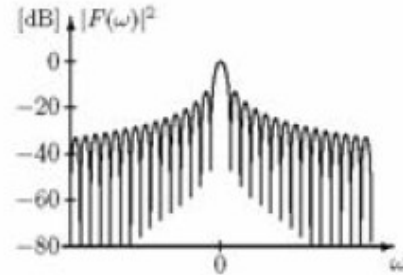
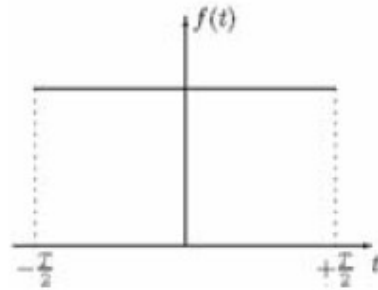
Josiah Willard Gibbs

(February 11, 1839 – April 28, 1903)

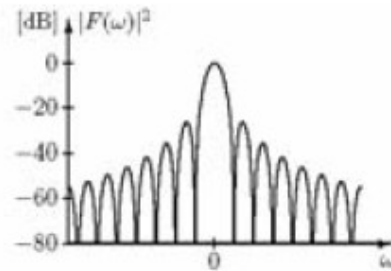
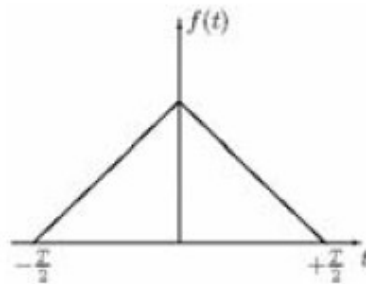
Gibbs' phenomenon live



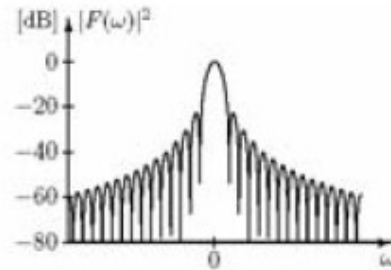
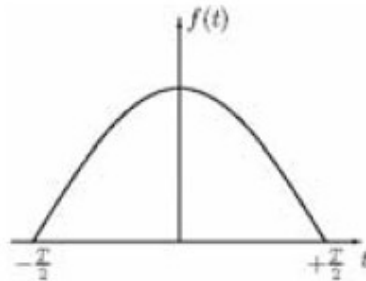
Windows I



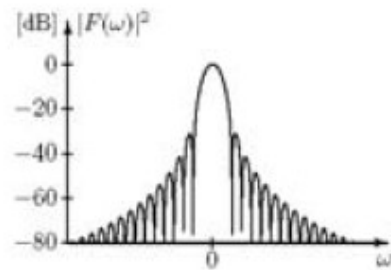
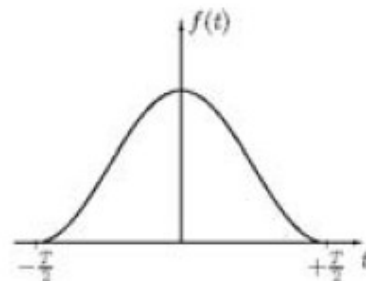
Rectangular window



Triangular window



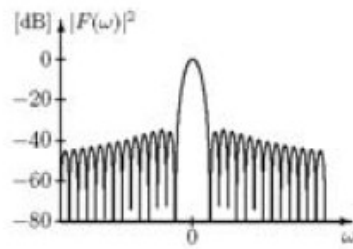
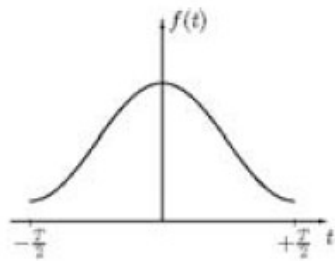
Cosine window



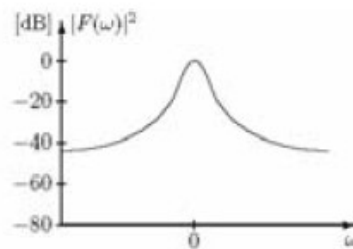
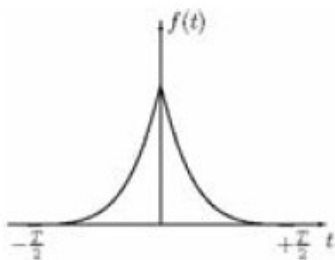
Hanning window



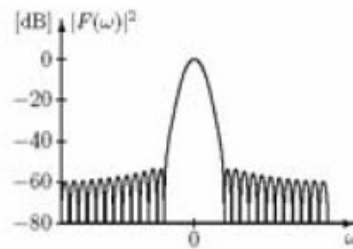
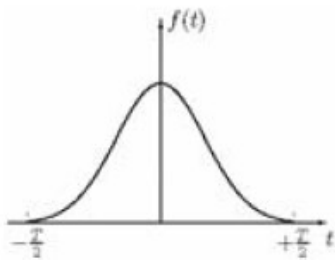
Windows II



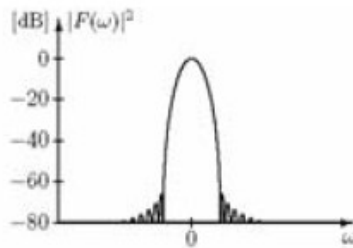
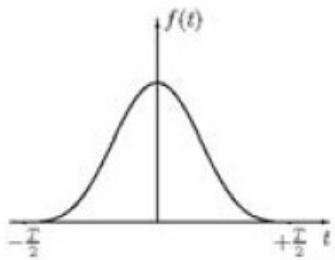
Hamming window



Triplet window

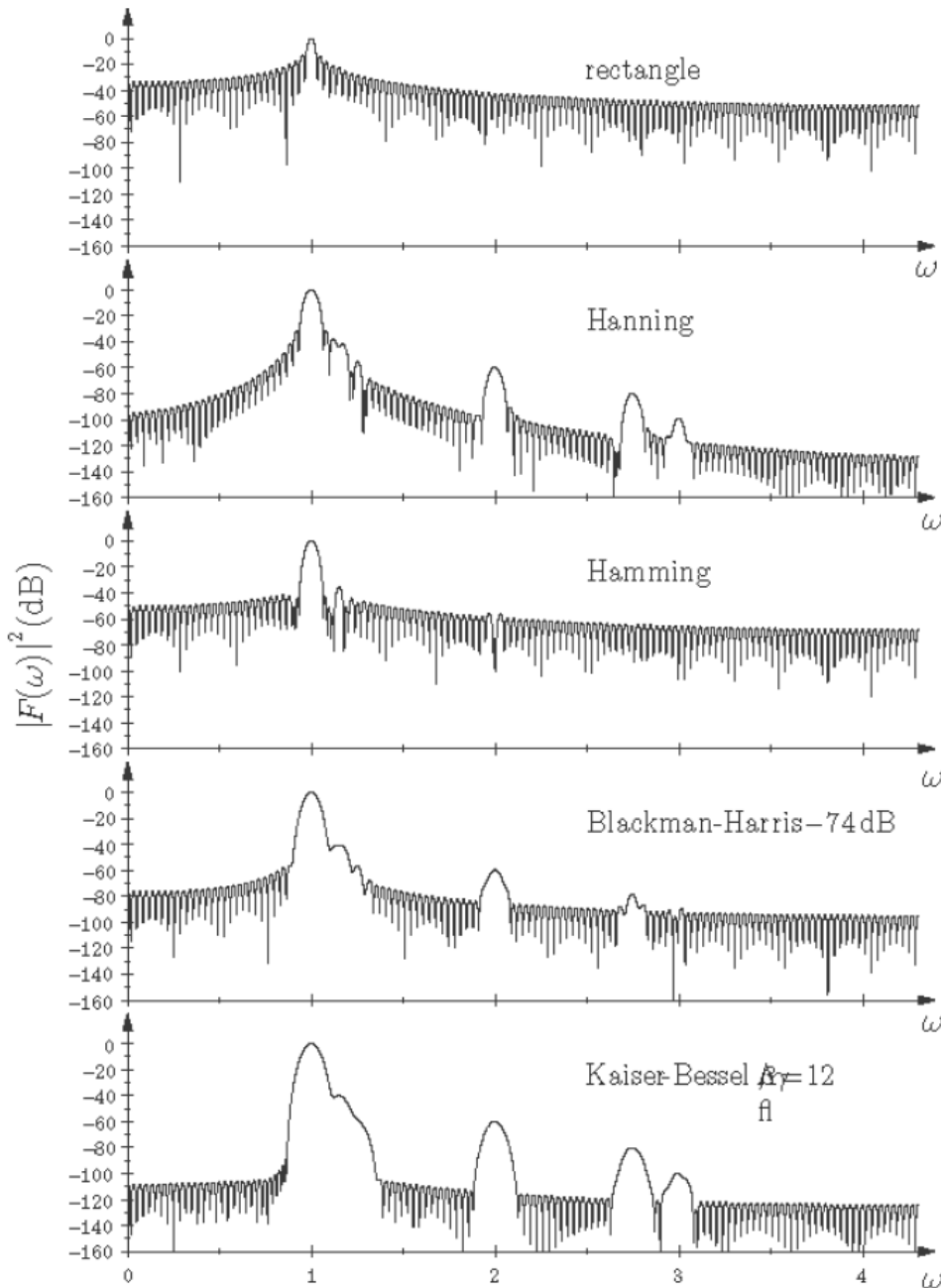


Gauss window



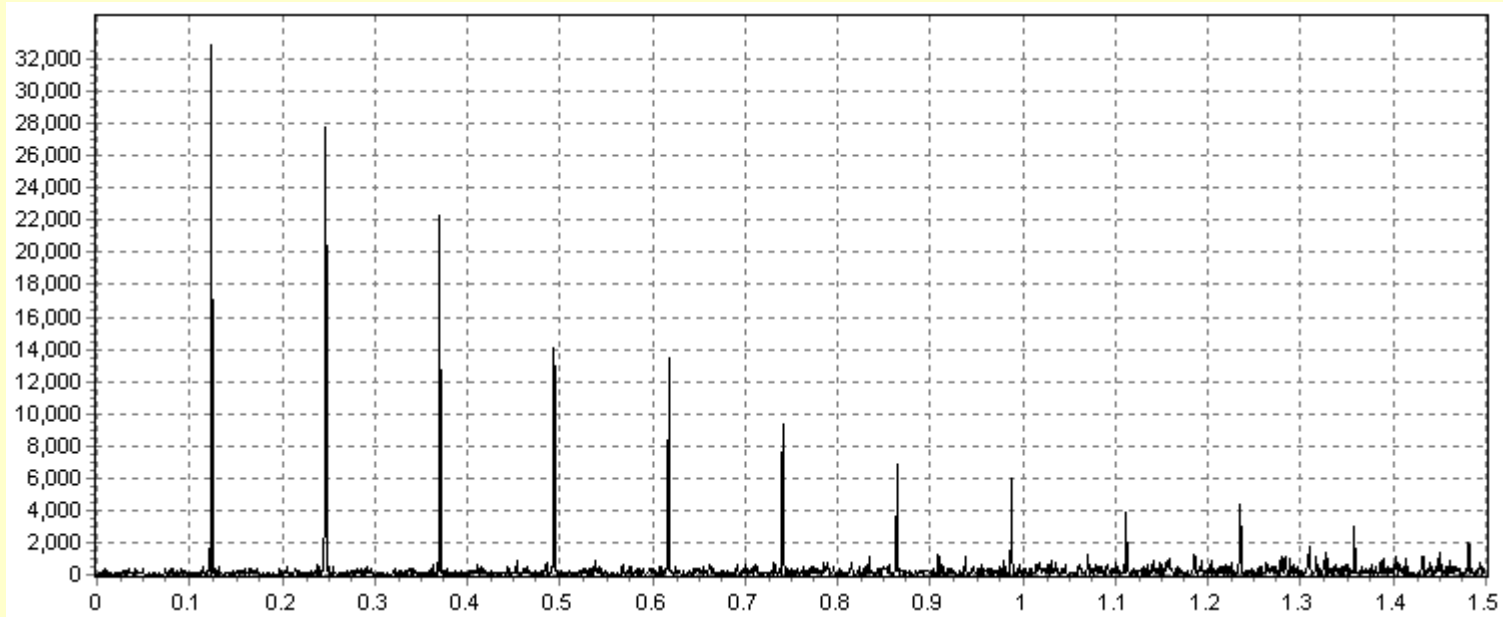
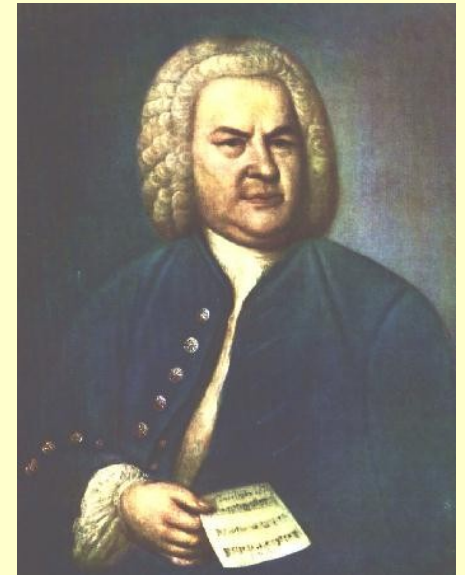
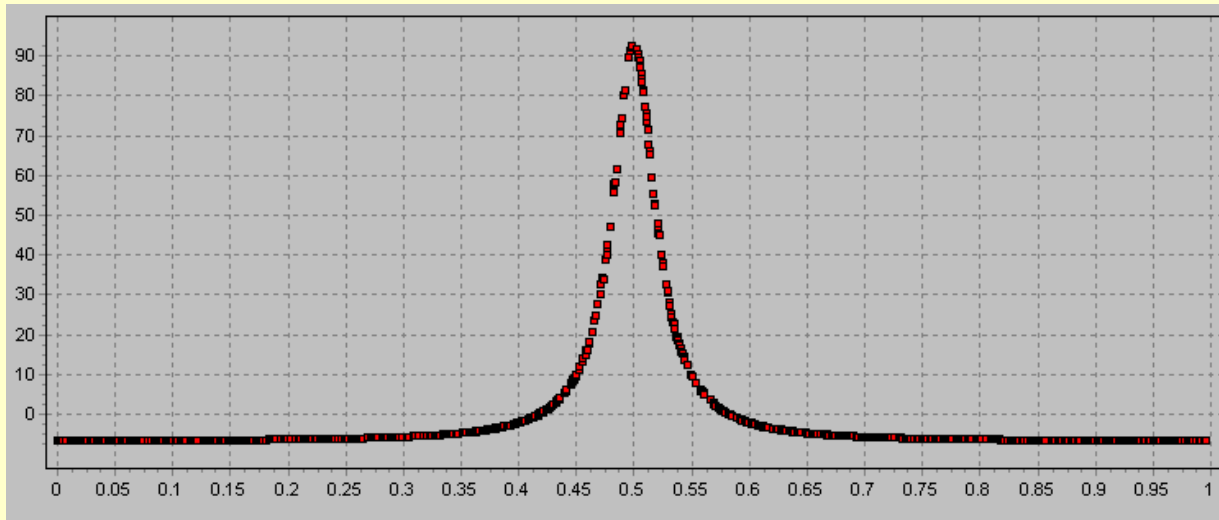
Kaiser-Bessel window

Use of windows



$$f(t) = \cos \omega t + 10^{-2} \cos 1.15 \omega t + 10^{-3} \cos 1.25 \omega t \\ + 10^{-3} \cos 2 \omega t + 10^{-4} \cos 2.75 \omega t + 10^{-5} \cos 3 \omega t$$

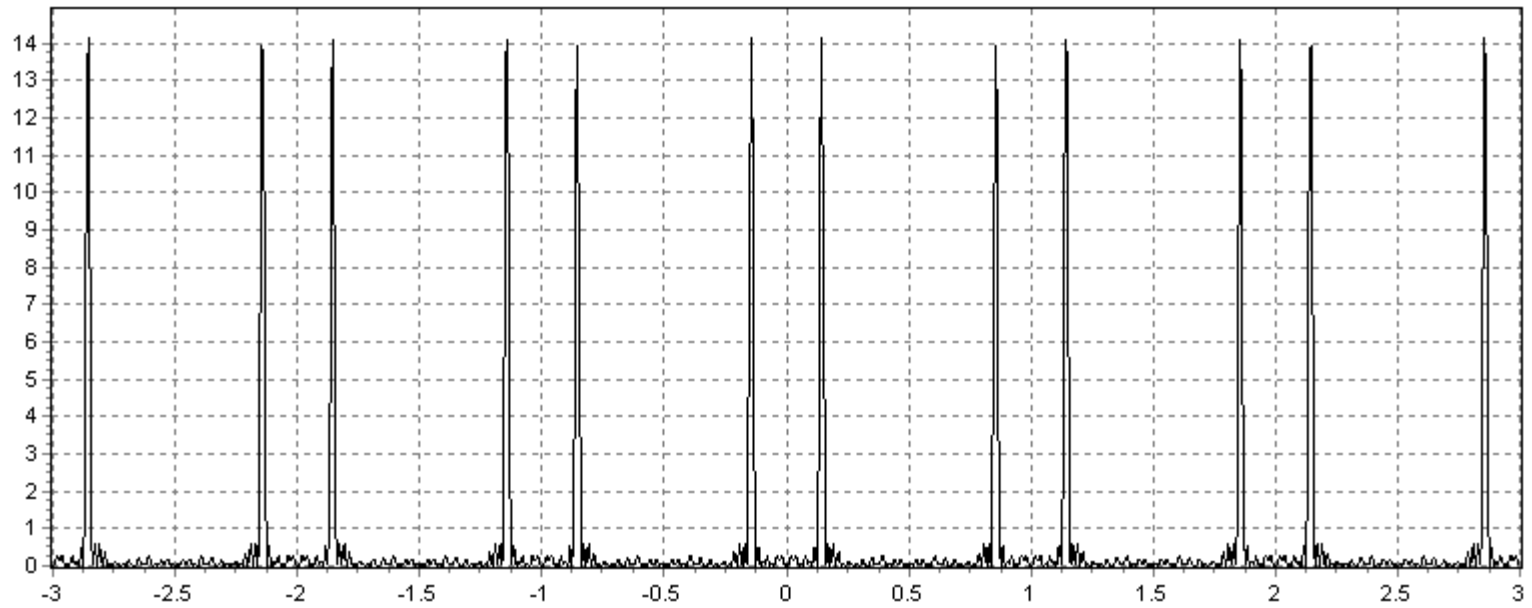
Signal is not harmonic



Nyquist Frequency

If T is your sampling rate which corresponds to a frequency of f_s , then signals with frequencies up to $f_s/2$ can be unambiguously reconstructed. This is the Nyquist frequency, N :

$$N < f_s/2$$



Comb function

In [mathematics](#), a **Dirac comb** (also known as an **impulse train** in [electrical engineering](#)) is a [periodic Schwartz distribution](#) constructed from [Dirac delta functions](#)

$$\Delta_T(t) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

for some given period T . Some authors, notably [Bracewell](#) as well as some textbooks authors in electrical engineering and circuit theory, refer to it as the **Shah function** (probably because its graph resembles the shape of the [Cyrillic](#) letter [sha](#) Ш). Because the Dirac comb function is periodic, it can be represented as a [Fourier series](#):

$$\Delta_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi nt/T}.$$

Transform of comb function

The **Fourier transform** of a Dirac comb is also a Dirac comb.

Unitary transform to ordinary frequency domain (Hz):

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{\mathcal{F}} \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) = \sum_{n=-\infty}^{\infty} e^{-i2\pi fnT}$$

Unitary transform to angular frequency domain (radians/sec):

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{\mathcal{F}} \frac{\sqrt{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{-i\omega nT}$$

Poisson sum

The sum $S(t)$ of a function $f(t)$ over all integers is equal to an equivalent summation of its continuous Fourier transform $\tilde{F}(\omega)$

$$S(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} f(t + nT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \tilde{F}(m\omega_0) e^{im\omega_0 t}$$

where the continuous Fourier transform is defined as

$$\tilde{F}(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

and the fundamental frequency ω_0 is

$$\omega_0 \stackrel{\text{def}}{=} \frac{2\pi}{T}.$$

An alternative definition of the continuous Fourier transform (namely, the unitary convention of mathematicians) and its corresponding Poisson summation formula are given below.