The Finnish Graduate School in Astronomy and Space Physics Summer School 2007:

Time Series Analysis

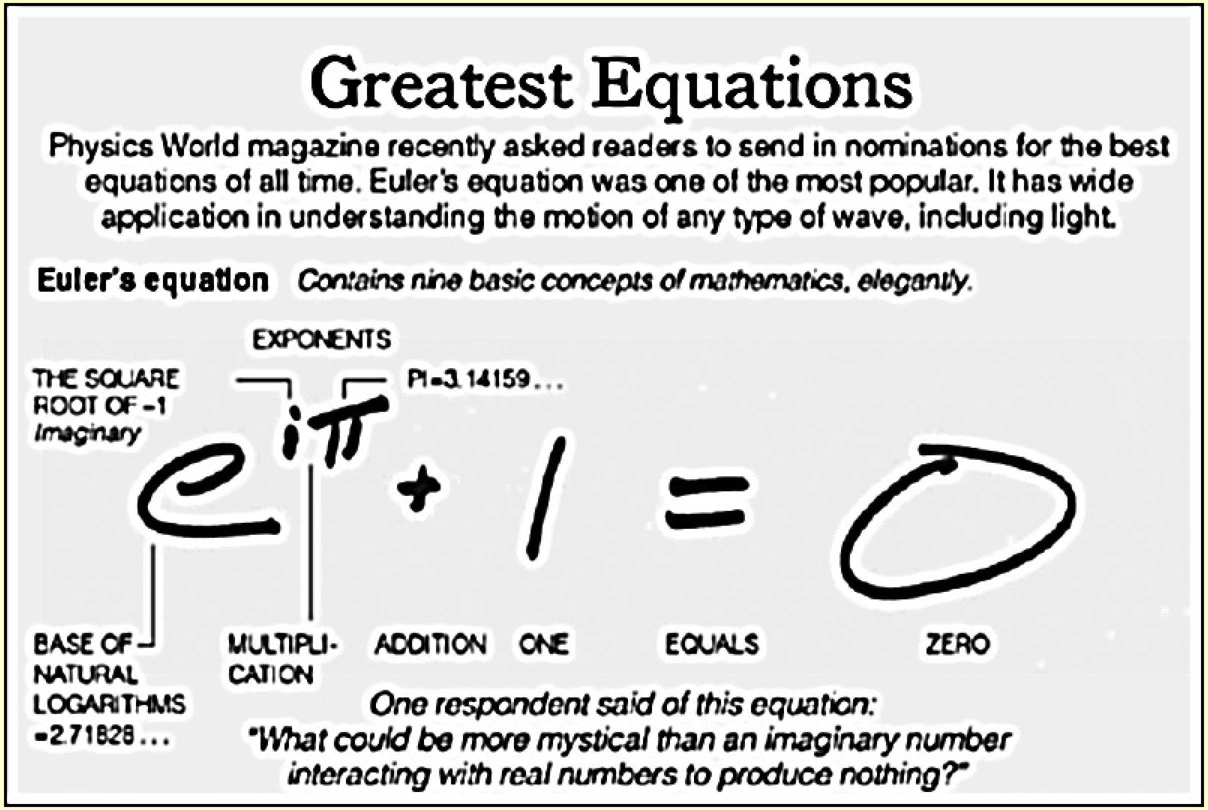
Part III. Methods

Numerical and statistical methods in time series analysis

Why is there so many peaks in spectra?

- Fourier transform is complex transform
- Data set is finite
- We can compute only approximately
- Data set is not equally spaced
- Signal is not harmonic
- The signal can contain more frequencies
- Signal parameters change in time
- Observations are noisy

Fourier transform is complex transform



Leonhard Paul Euler
April 15, 1707
September 18 [O.S. September 7] 1783

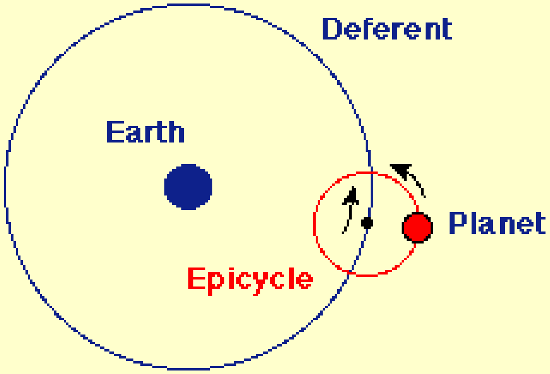
But harmonics are real!

Claudius Ptolemaeus (Ptolemy)
ca. 90 – ca. 168 AD

Abstract clock $= C(\nu)$

Schema huius præmissæ diuisionis Sphærarum.

- Clock
- Cisoid (cosinusoid + sinusoid)
- Complex harmonic
- Circle (of unit radius)
- Circular harmonic
- Cycle
- etc.



The most important algebraic property of abstract clocks

$$C(\nu_1)C(\nu_2) = C(\nu_1 + \nu_2)$$

$$C(\nu)C(-\nu) = C(0)$$

Just clock which is not working (rotating) at all

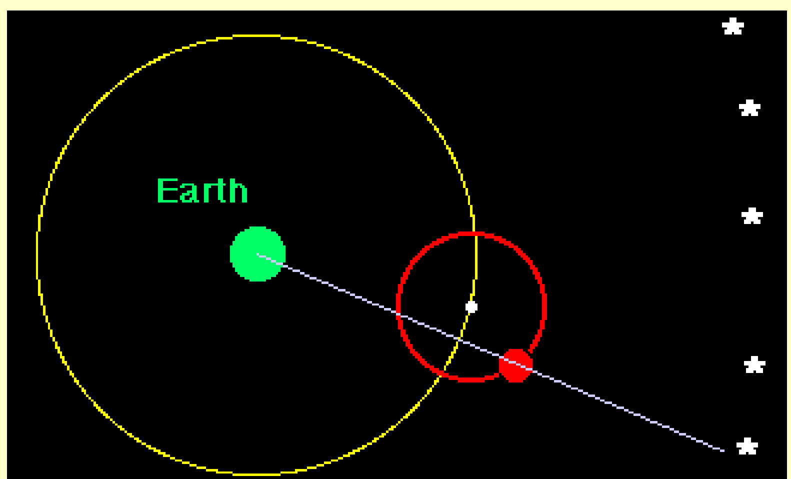
Phases and rotators

$$\text{Phase} = p(\phi)$$

$$p(\phi_1)p(\phi_2) = p(\phi_1 + \phi_2)$$

$$\text{Rotator} = Ap(\phi)C(\nu)$$

$$\text{Orbit} = A^{\text{def}}p(\phi^{\text{def}})C(\nu^{\text{def}}) + A^{\text{cyc}}p(\phi^{\text{cyc}})C(\nu^{\text{cyc}})$$



Definition of trigonometric functions

$$\cos(\nu) = \frac{C(\nu) + C(-\nu)}{2}$$

Similarly

$$\frac{C(\nu) - C(-\nu)}{2}$$

Using "back-rotation"

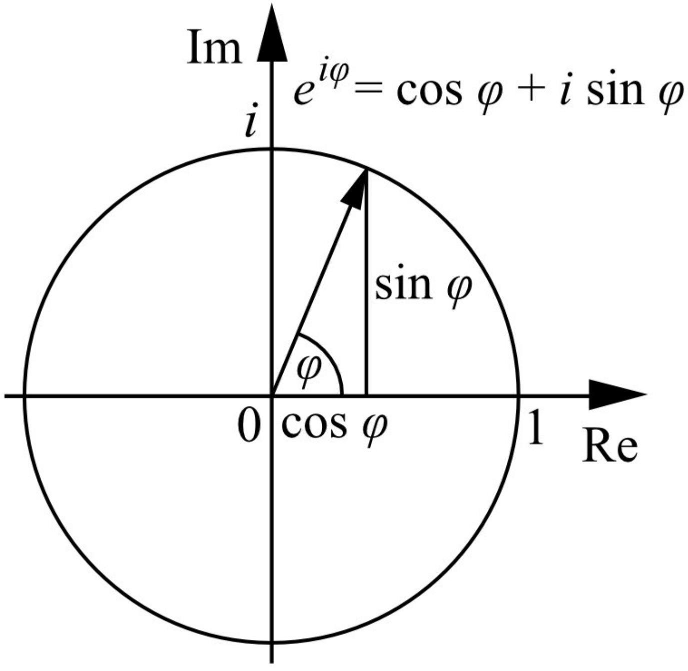
$$j^4 = p(0)$$

$$\sin(\nu) = \frac{C(\nu) - C(-\nu)}{2j}$$

Euler's formula again

$$e^{j2\pi\nu t} = \cos(2\pi\nu t) + j\sin(2\pi\nu t)$$

- Natural number 2.
- Real number ν.
- Two transcendental numbers e and π.
- Imaginary number j.
- Exponentiation.
- Two trigonometric functions.



The Fourier Transform and its Inverse in another format

$$F(\omega) = f(t)\exp(-i\omega t)\,dt$$

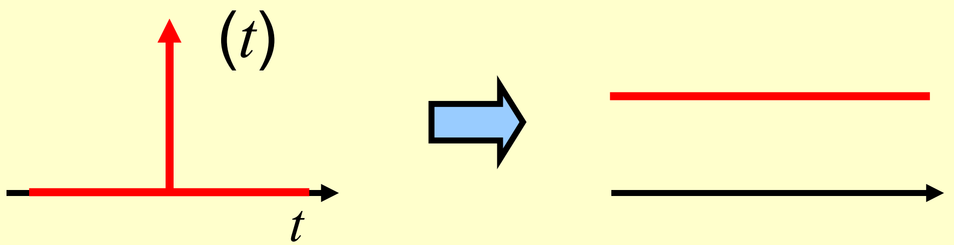
$-$

$$f(t) = \frac{1}{2\pi} F(\omega)\exp(i\omega t)\,d\omega$$

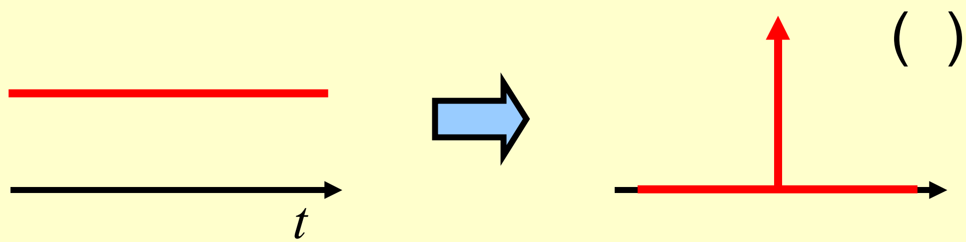
$-$

The Fourier Transform of $\delta(t)$ is 1

$$\delta(t)\,\exp(-i\omega t)\,dt = \exp(-i\omega[0]) = 1$$



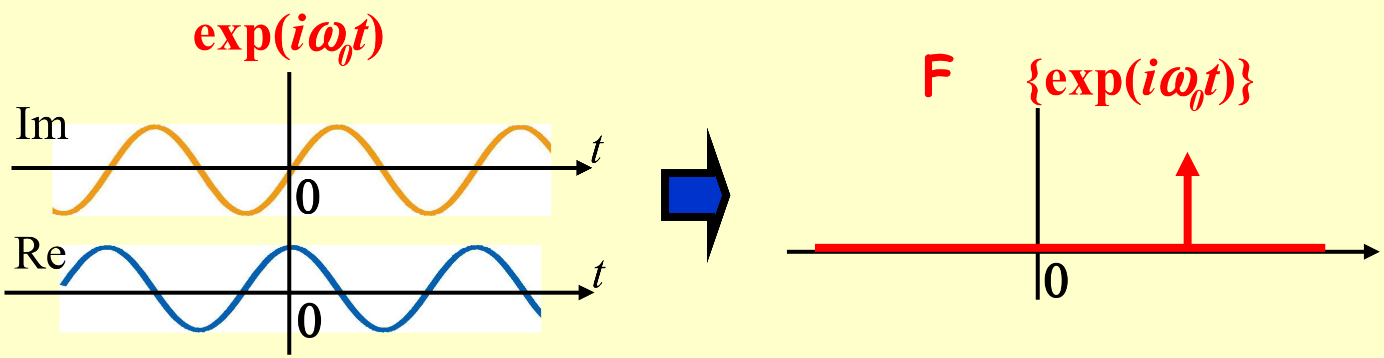
And the Fourier Transform of 1 is ( ): $$1\,\exp(-i\omega t)\,dt = 2\pi\,\delta(\omega)$$



The Fourier transform of $\exp(i\omega_0 t)$

$$\mathcal{F}\left\{\exp(i\omega_0 t)\right\} = \int_{-\infty}^{\infty} \exp(i\omega_0 t)\exp(-i\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} \exp(-i[\omega-\omega_0]t)\,dt = 2\pi\,\delta(\omega-\omega_0)$$



The function $\exp(i\omega_0 t)$ is the essential component of Fourier analysis.

It is a pure frequency.

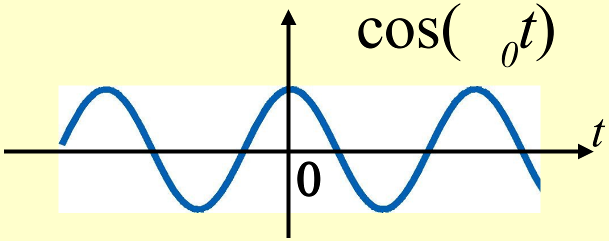
The Fourier transform of $\cos(\omega_0\, t)$

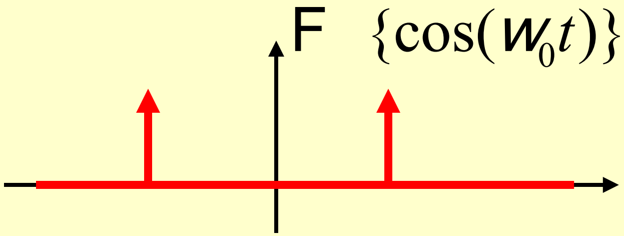
$$\mathcal{F}\{\cos(\omega_0 t)\} = \int_{-\infty}^{\infty} \cos(\omega_0 t)\exp(-i\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[\exp(i\omega_0 t) + \exp(-i\omega_0 t)\right]\exp(-i\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega-\omega_0]t)\,dt + \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega+\omega_0]t)\,dt$$

$$= \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$





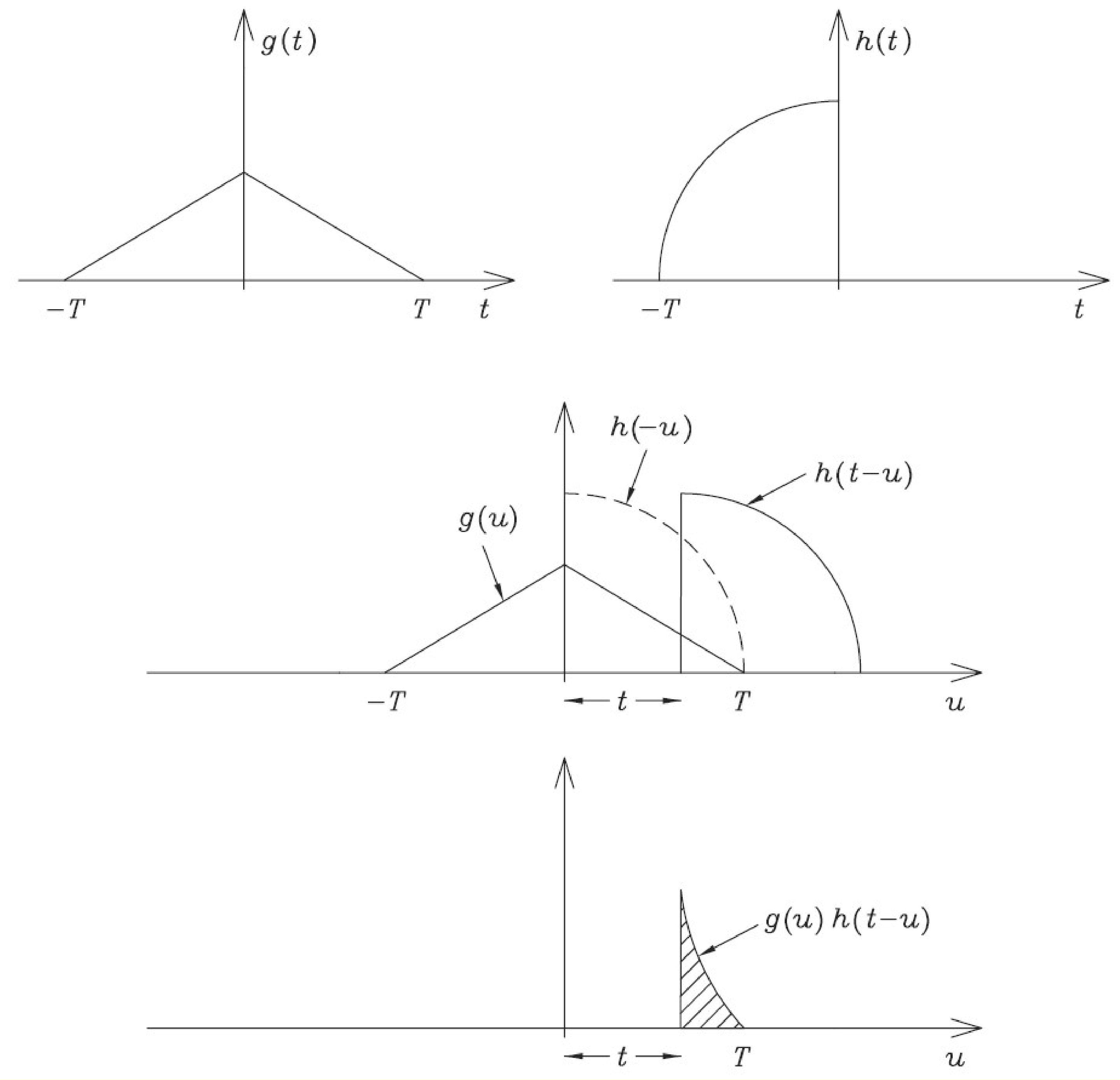
Convolution

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u) g(x-u) du$$

If

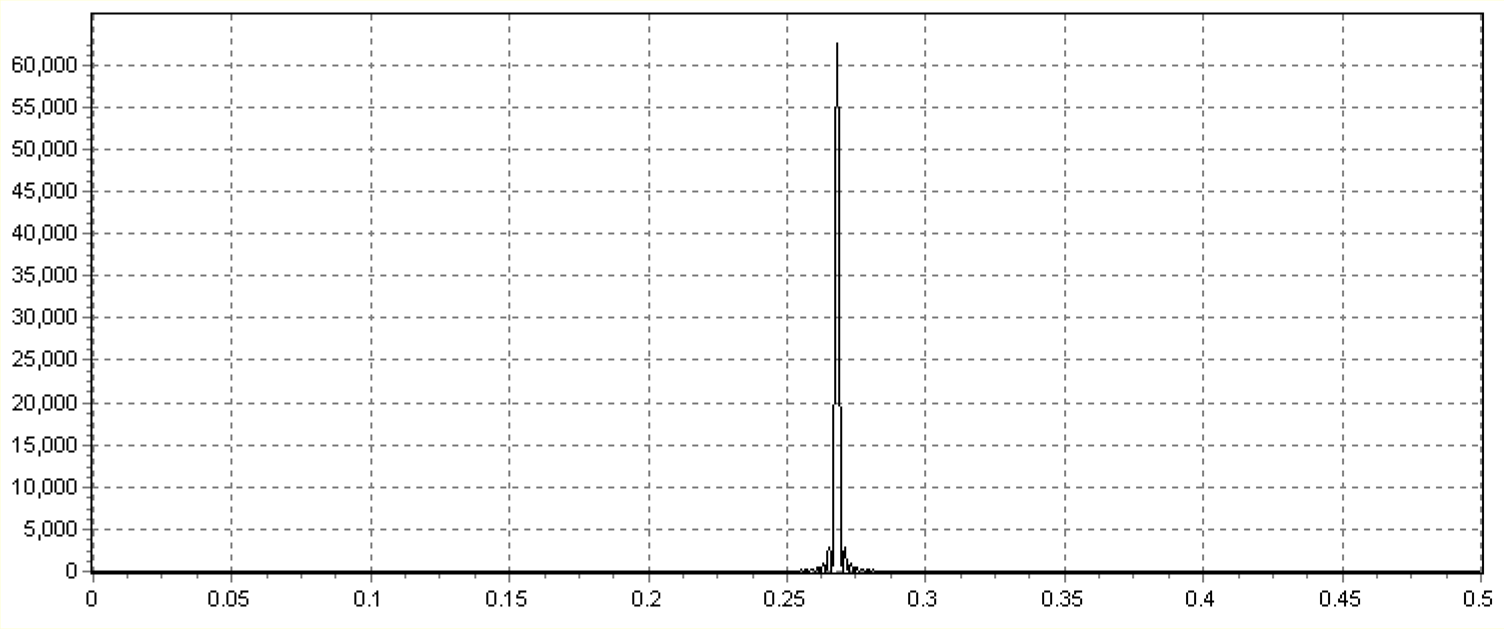
$$\mathcal{F}(f(t)) = F(\nu) \text{ and } \mathcal{F}(g(t)) = G(\nu)$$

$$\mathcal{F}(f(t) * g(t)) = F(\nu) G(\nu)$$

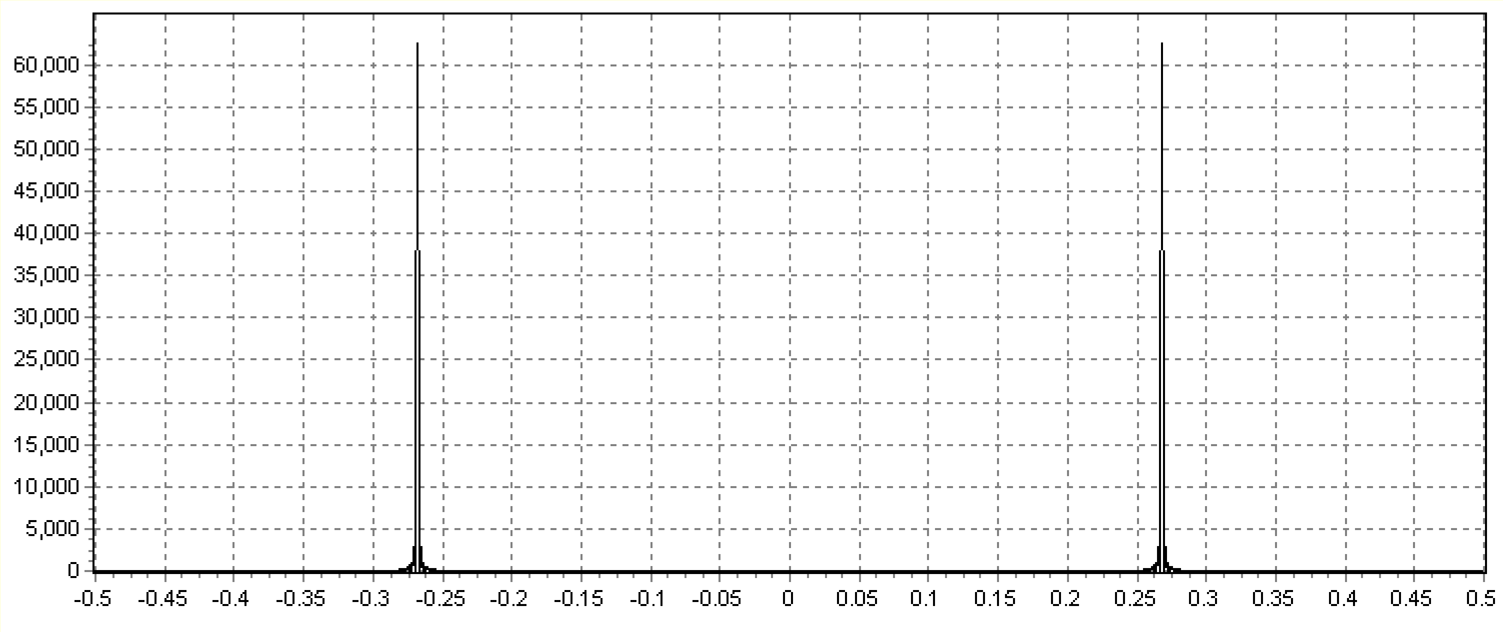


How Convolution works?

Ideal case, one component, one peak

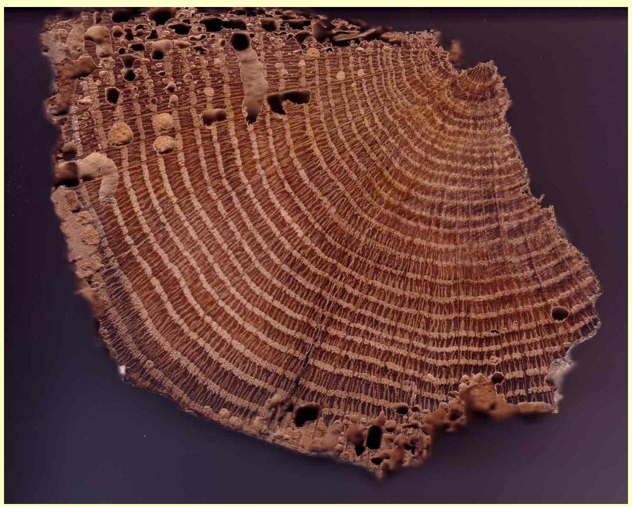


Fourier transform is complex, harmonics are real

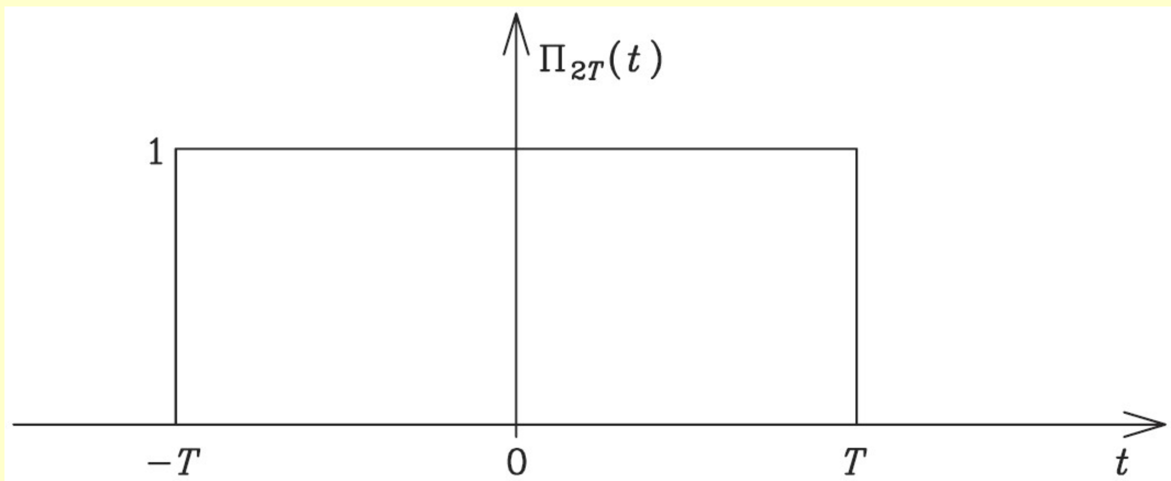


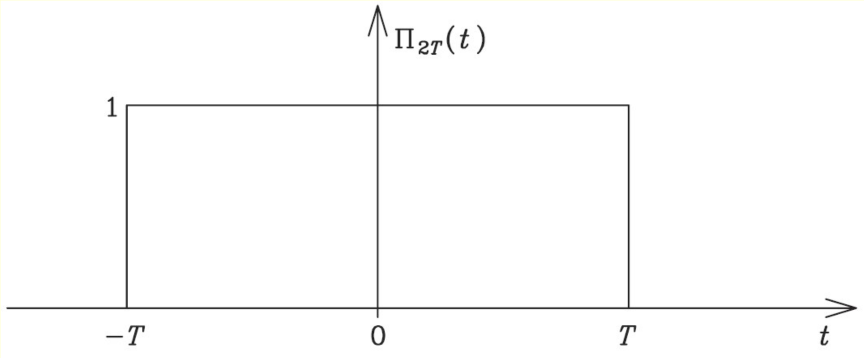
$$\cos(\nu) = \frac{C(\nu) + C(-\nu)}{2}$$

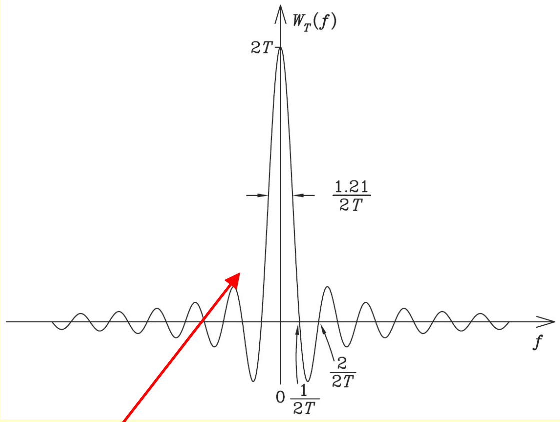
Data sets are finite



$$\Pi(x) = \left\{ \begin{array}{ll} 0, & |x| > \frac{1}{2} \\ \frac{1}{2}, & |x| = \frac{1}{2} \\ 1, & |x| < \frac{1}{2} \end{array} \right\}$$







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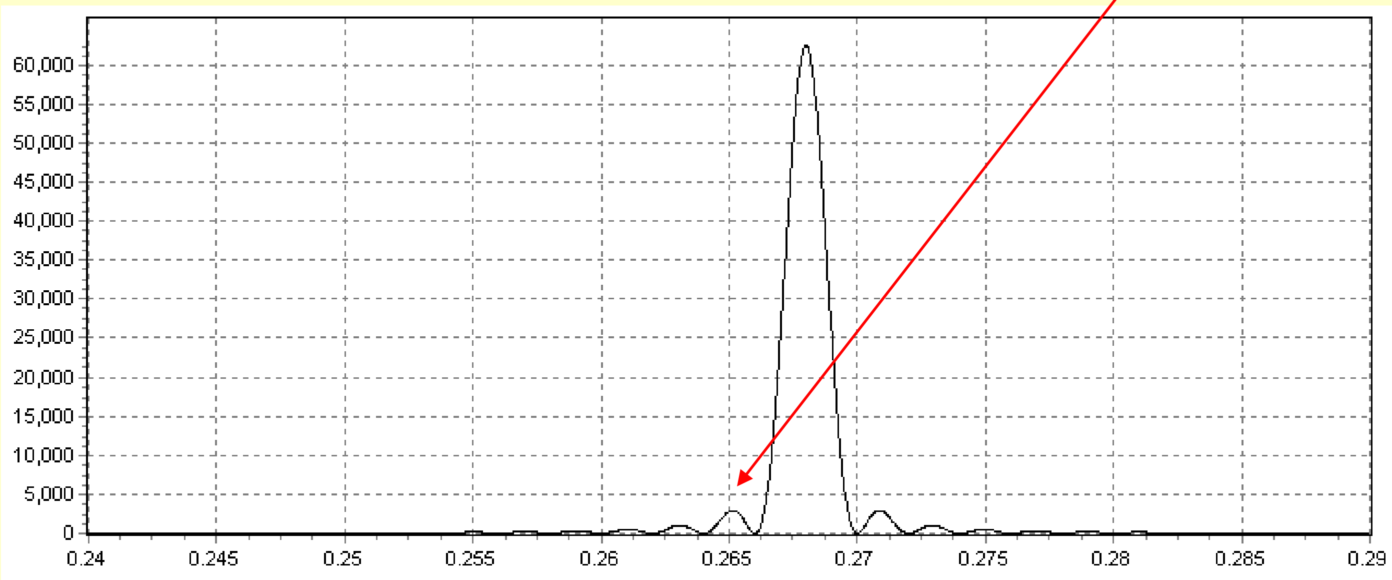
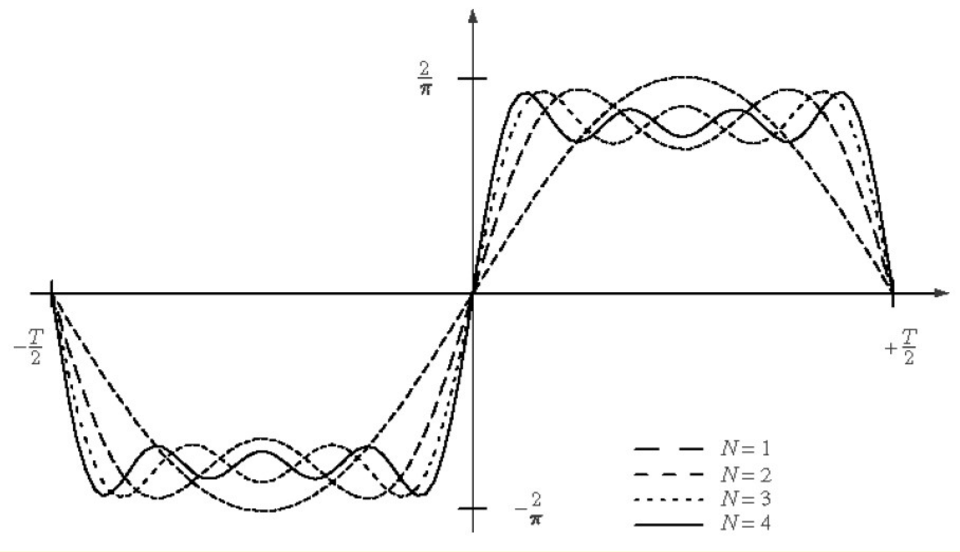
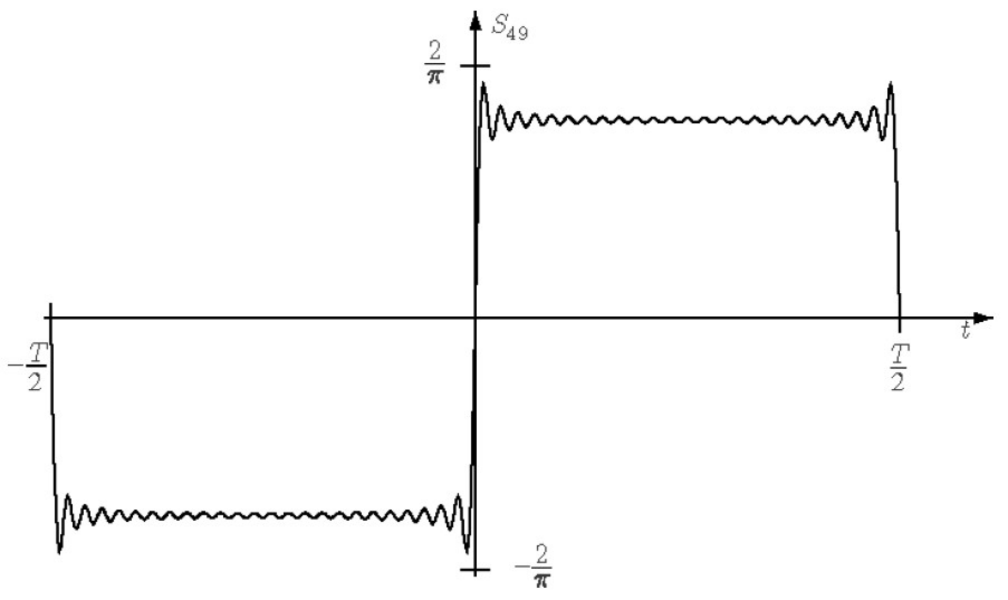
$$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$$



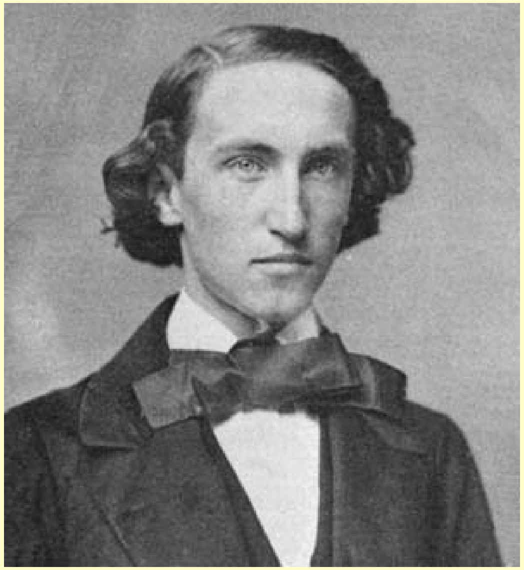
More peaks

We can compute only approximately





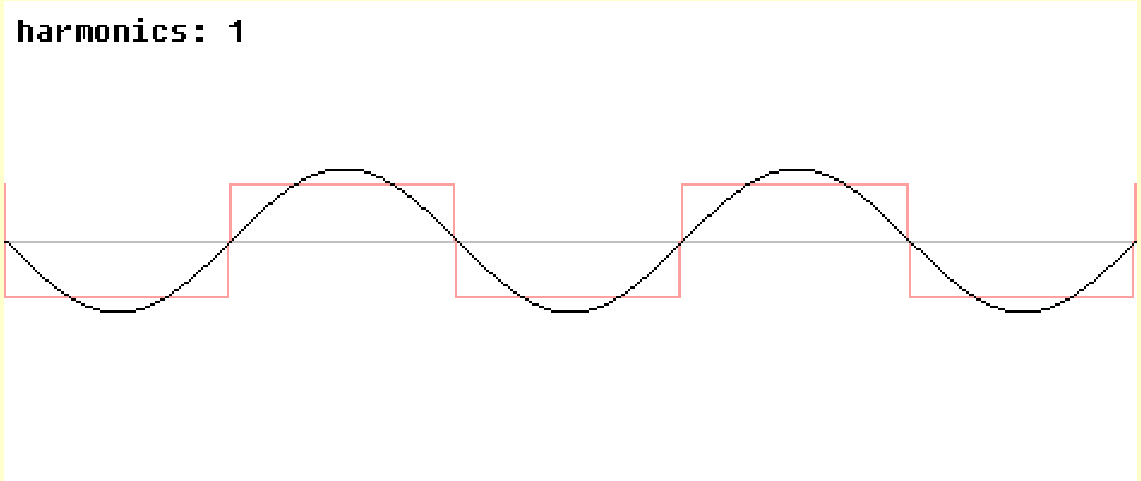
- Gibbs' phenomenon
- Gibbs' overshoot



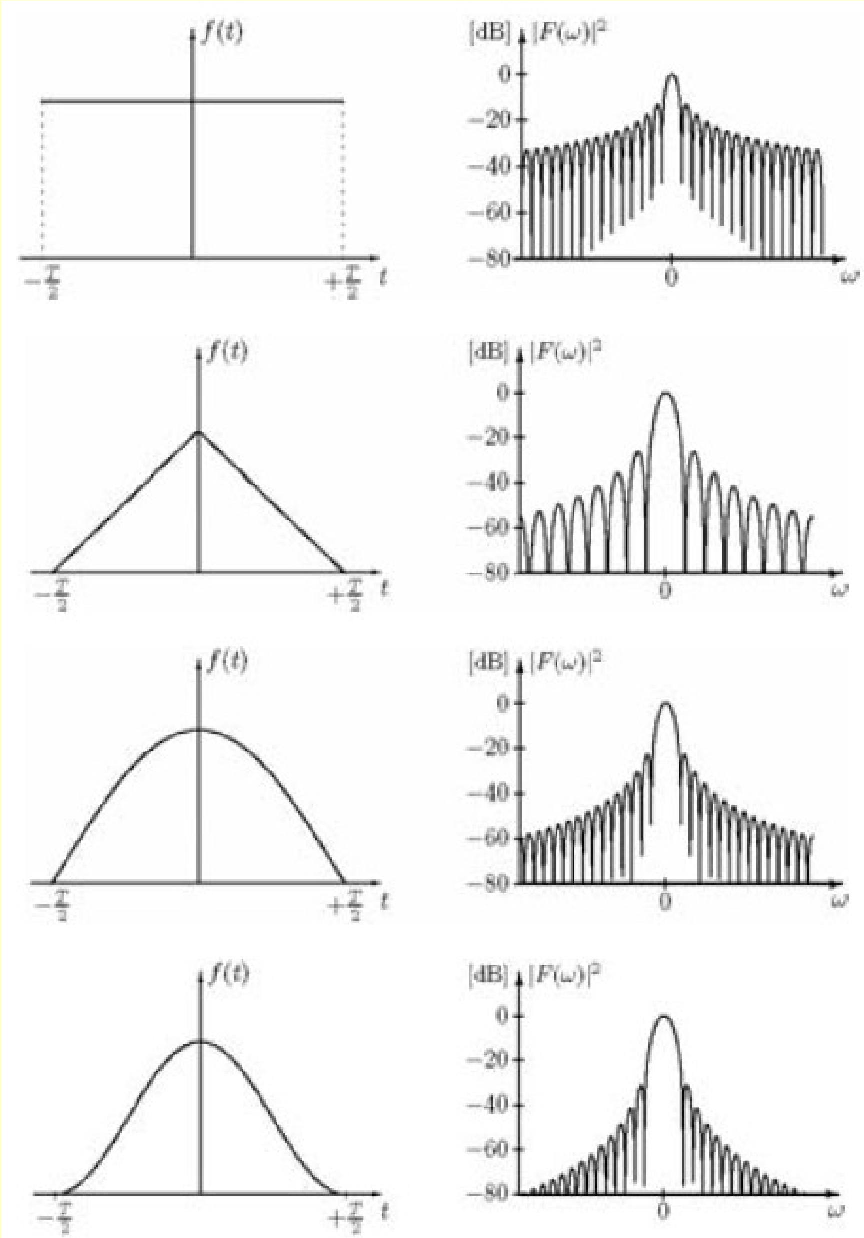
Josiah Willard Gibbs

(February 11, 1839 – April 28, 1903)

Gibbs' phenomenon live



Windows I



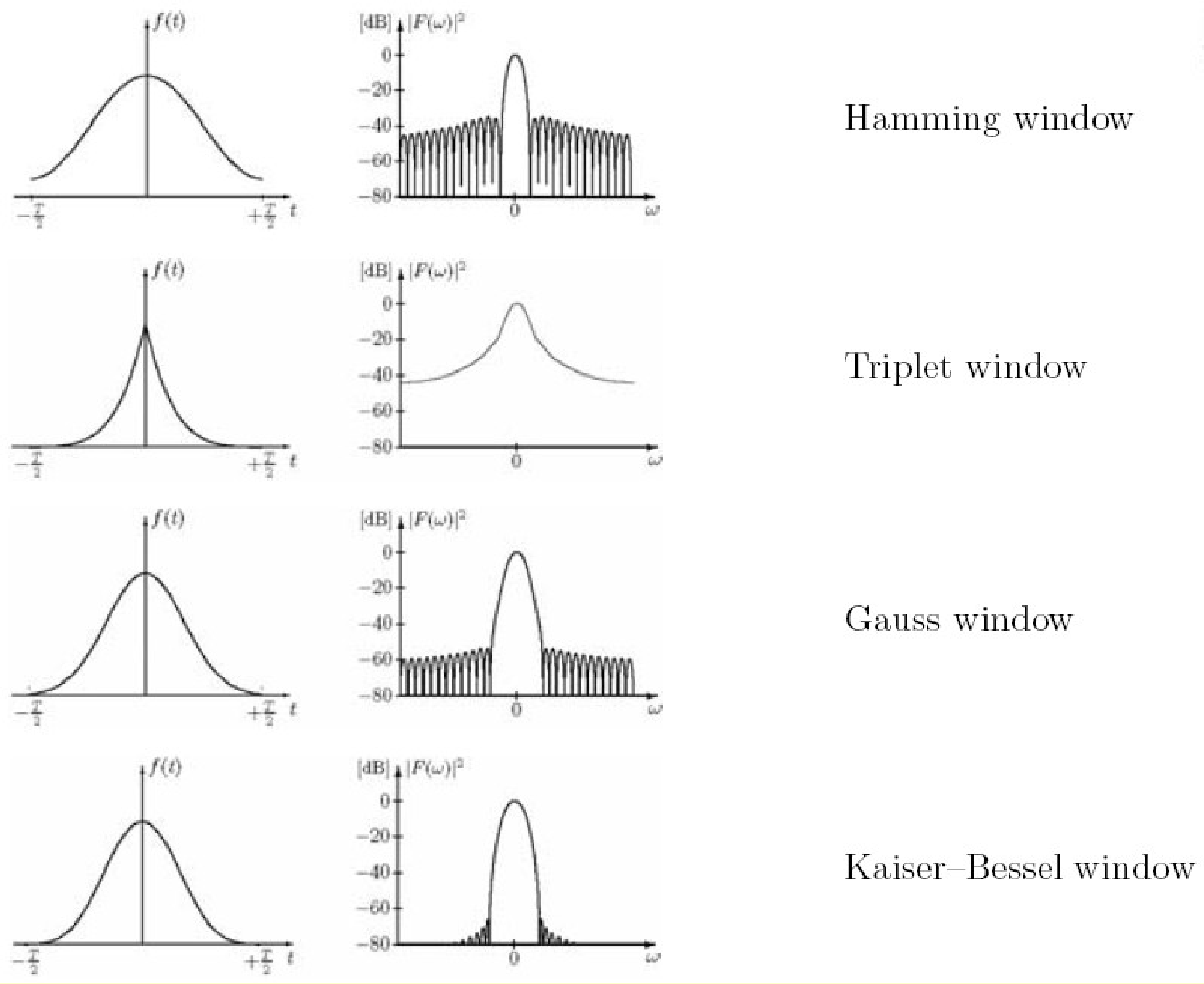
Rectangular window

Triangular window

Cosine window

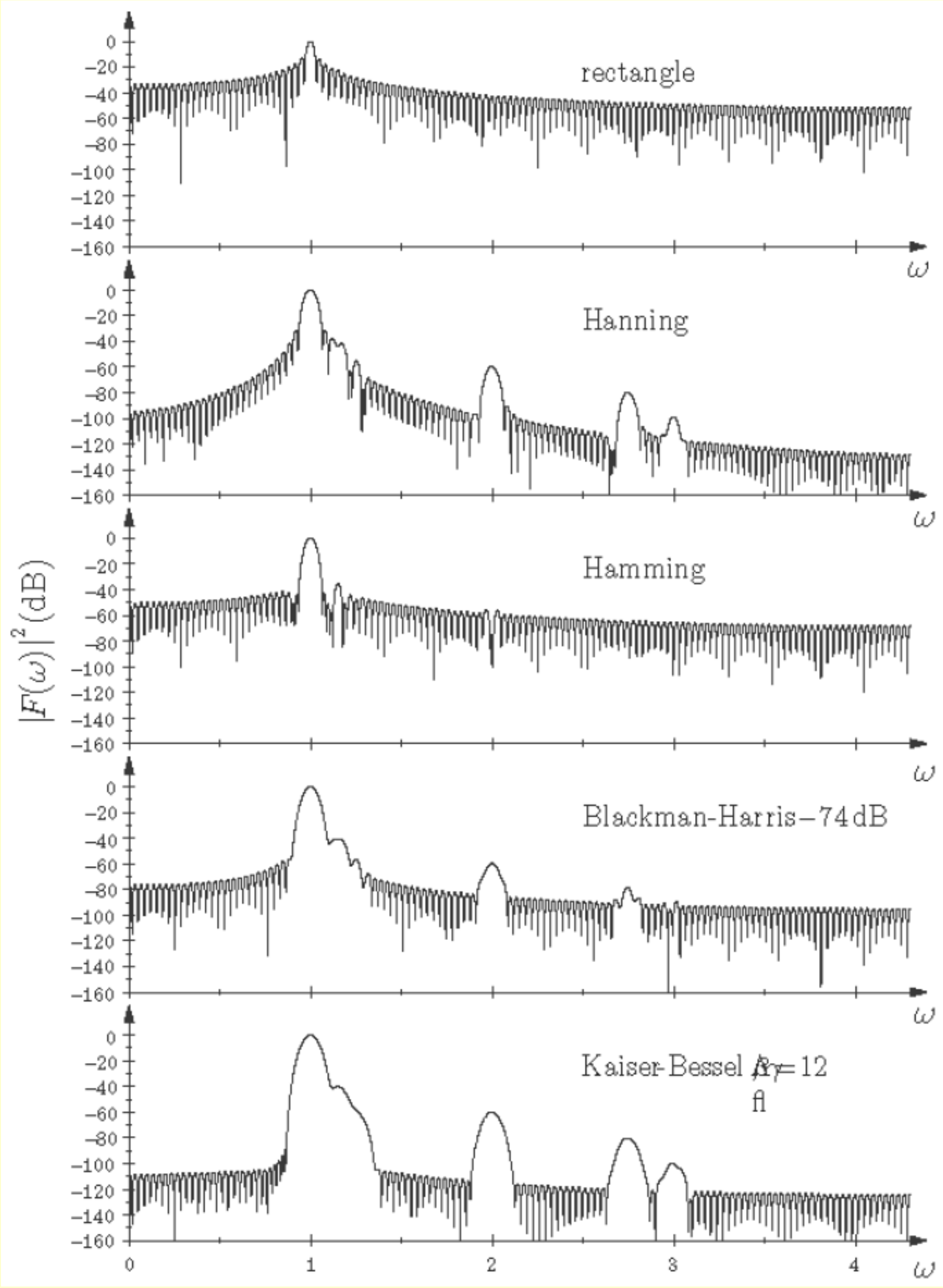
Hanning window

Windows II



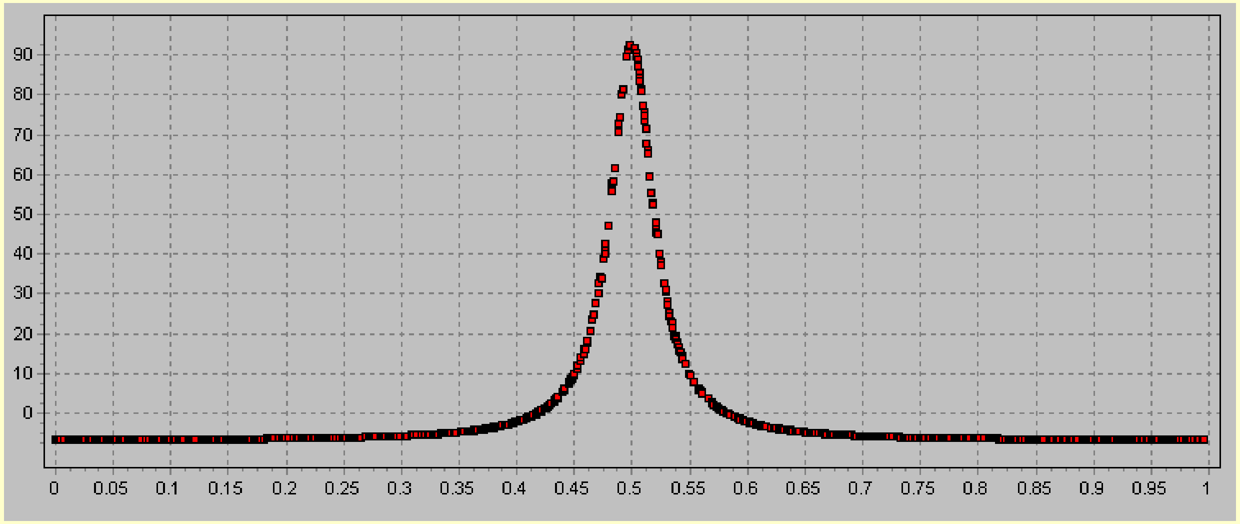
Designed for Windows® XP

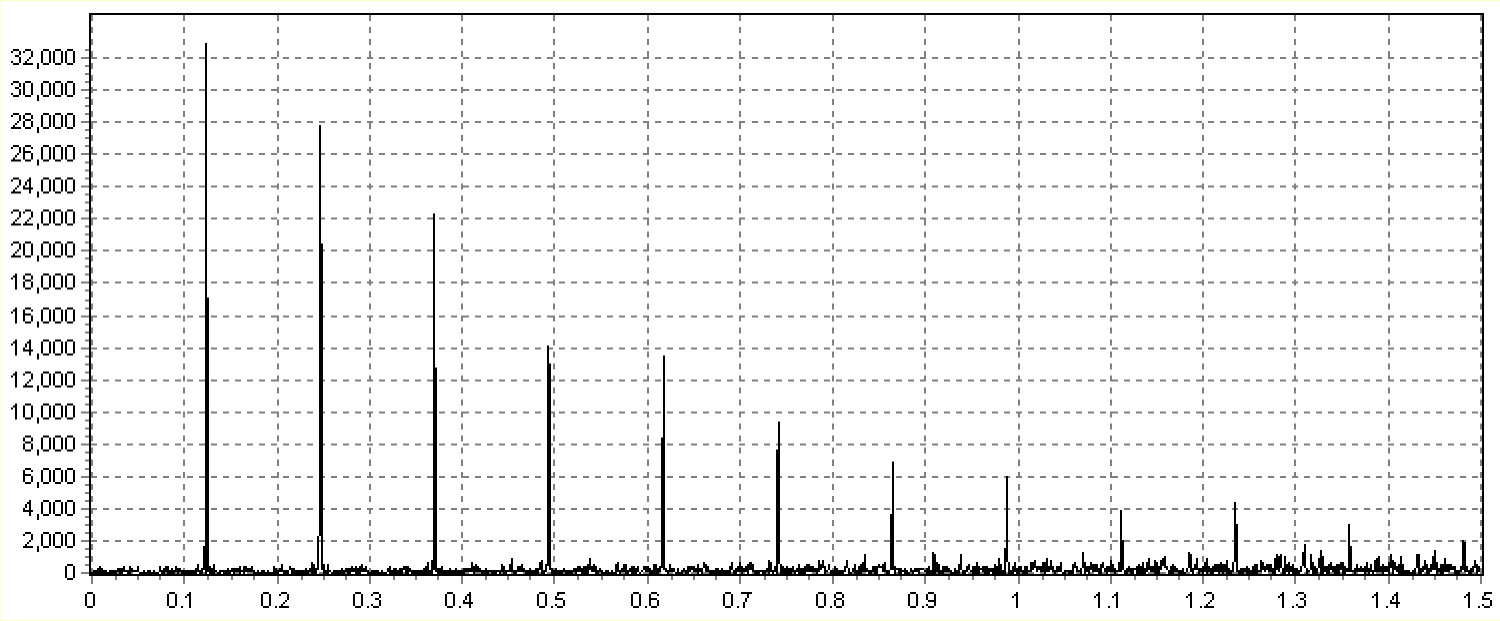
Use of windows



$$f(t) = \cos\omega t + 10^{-2}\cos 1.15\omega t + 10^{-3}\cos 1.25\omega t + 10^{-3}\cos 2\omega t + 10^{-4}\cos 2.75\omega t + 10^{-5}\cos 3\omega t$$

Signal is not harmonic

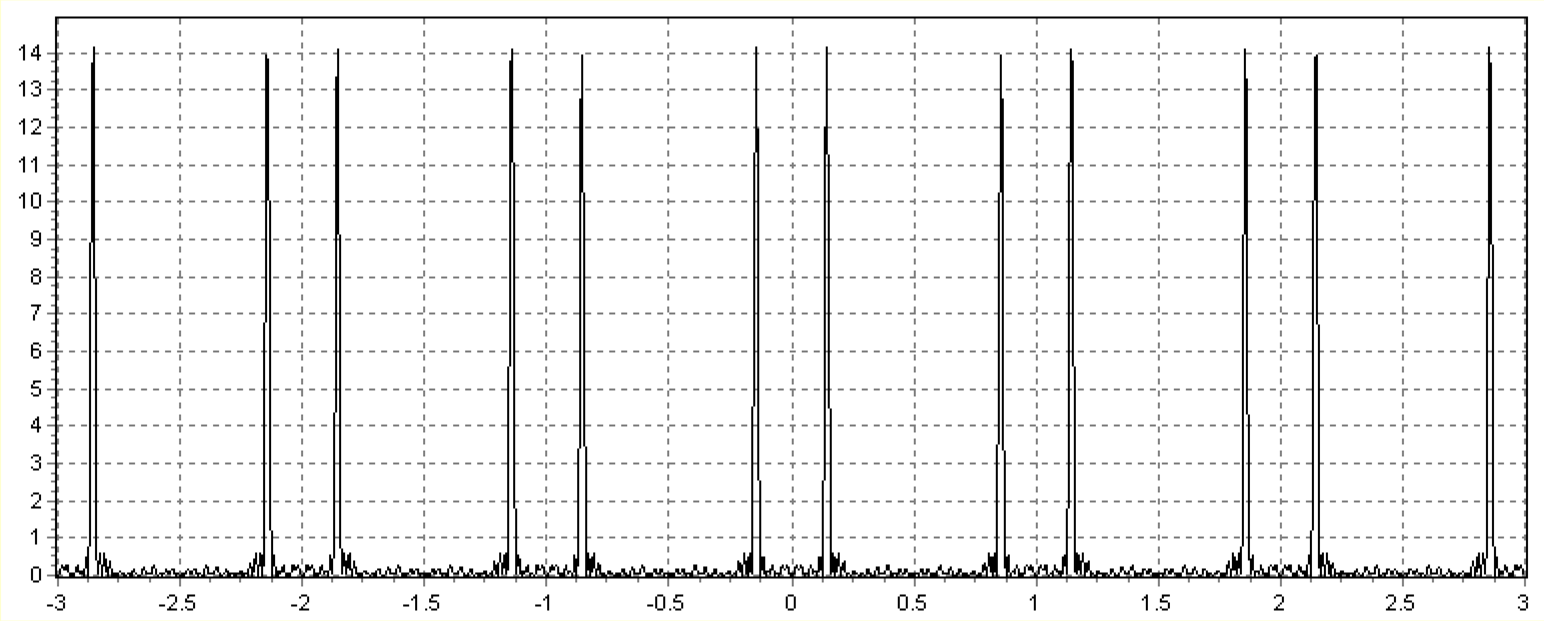




Nyquist Frequency

If *T* is your sampling rate which corresponds to a frequency of f_s, then signals with frequencies up to $f_s/2$ can be unambiguously reconstructed. This is the Nyquist frequency, N:

$$N < f_s/2$$



Comb function

In mathematics, a **Dirac comb** (also known as an **impulse train** in electrical engineering) is a periodic Schwartz distribution constructed from Dirac delta functions

$$\Delta_T(t) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

for some given period *T*. Some authors, notably Bracewell as well as some textbooks authors in electrical engineering and circuit theory, refer to it as the **Shah function** (probably because its graph resembles the shape of the Cyrillic letter sha Ш). Because the Dirac comb function is periodic, it can be represented as a Fourier series:

$$\Delta_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi nt/T}.$$

Transform of comb function

The Fourier transform of a Dirac comb is also a Dirac comb.

Unitary transform to ordinary frequency domain (Hz):

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \quad \stackrel{\mathcal{F}}{\Longleftrightarrow} \quad \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f-\frac{k}{T}\right) \quad = \sum_{n=-\infty}^{\infty} e^{-i2\pi f n T}$$

Unitary transform to angular frequency domain (radians/sec):

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \quad \stackrel{\mathcal{F}}{\Longleftrightarrow} \quad \frac{\sqrt{2\pi}}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega-k\frac{2\pi}{T}\right) \quad = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} e^{-i\omega n T}$$

Poisson sum

The sum $S(t)$ of a function $f(t)$ over all integers is equal to an equivalent summation of its continuous Fourier transform $\tilde{F}(\omega)$

$$S(t) \stackrel{\mathrm{def}}{=} \sum_{n=-\infty}^{\infty} f(t+nT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \tilde{F}(m\omega_0)\ e^{im\omega_0 t}$$

where the continuous Fourier transform is defined as

$$\tilde{F}(\omega) \stackrel{\mathrm{def}}{=} \int_{-\infty}^{\infty} f(t)\ e^{-i\omega t} dt$$

and the fundamental frequency ω_0 is

$$\omega_0 \stackrel{\mathrm{def}}{=} \frac{2\pi}{T}.$$

An alternative definition of the continuous Fourier transform (namely, the unitary convention of mathematicians) and its corresponding Poisson summation formula are given below.