# Plasma Theory

Introductory lecture at the Summer School of the
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# Task: Describe plasma theory in 2 hours

Impossible? No!

Compare with general relativity

$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ and that's it!

But if you want to describe the dynamics of two colliding rotating black holes, well, that is another matter.

One can claim that all plasma physics is described by Boltzmann's equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = \left( \frac{\partial f}{\partial t} \right)_c$$

With this only you do not get far to solve any relevant plasma physics problems

What does Boltzmann's equation tell about plasma physics?

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_c$$

distribution function: plasma physics is statistical physics

role of EM forces: plasma physics is electrodynamics

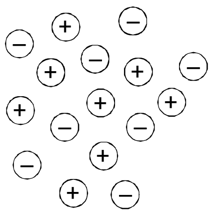
collisions can break the conservation of phase space density

How to define the plasma state?

**Plasma is *quasi-neutral ionized gas* containing enough *free charges* to make *collective electromagnetic effects* important for its physical behaviour.**

The most fundamental plasma properties are
- Debye screening
- plasma oscillations
- gyro motion of plasma particles

# Debye screening



Coulomb potential of each charge: $\varphi = \frac{q}{4\pi\epsilon_0 r}$

Assume thermal equilibrium (Boltzmann distribution)

$$n_\alpha(\mathbf{r}) = n_{0\alpha} \exp\left(-\frac{q_\alpha \varphi}{k_B T_\alpha}\right)$$

$\alpha$ labels the particle populations (e.g., $e, p$)

Introduce a test charge $q_T$. What will be its potential?

Home exercise: $\varphi = \frac{q_T}{4\pi\epsilon_0 r} \exp\left(-\frac{r}{\lambda_D}\right)$ ; $\lambda_D^{-2} = \frac{1}{\epsilon_0} \sum_\alpha \frac{n_{0\alpha} q_\alpha^2}{k_B T_\alpha}$

Debye length: $\lambda_D \propto \sqrt{\frac{T}{n}}$

Number of particles in a Debye sphere: $N = \frac{4\pi}{3} n_0 \lambda_D^3$

A little better(?) definition for plasma

$$\frac{1}{\sqrt[3]{n_0}} \ll \lambda_D \ll L$$

$L$ is the size of the system

Plasma parameter: $\Lambda = n_0 \lambda_D^3 \gg 1$

# Plasma oscillation

Assume: $n_0$ fixed ions (+) & $n_0$ moving electrons (–)

Apply a small electric field $\mathbf{E}_1$

→ electrons move:

$$\begin{aligned} n_+ &= n_0 \\ n_- &= n_0 + n_1(\mathbf{r},t) \;\; ; \;\; n_1 \ll n_0 \end{aligned}$$

Electron continuity equation: $\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$

$n = n_0 + n_1(\mathbf{r},t) \;\; ; \mathbf{u} = \underbrace{\cancel{\mathbf{u}_0}}_{0} + \mathbf{u}_1(\mathbf{r},t)$ ( $\mathbf{u}_0 = 0 \leftrightarrow$ electrons are assumed cold )

$$\underbrace{\frac{\partial \cancel{n_0}}{\partial t}}_{0} + \frac{\partial n_1}{\partial t} + \nabla \cdot ((n_0 + n_1)\mathbf{u}_1) = 0 \;\; \Rightarrow \;\; \frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \mathbf{u}_1 + \nabla \cdot \underbrace{(\cancel{n_1 \mathbf{u}_1})}_{0 \text{ 2nd order}} = 0$$

Linearized continuity equation (1st order terms only): $\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \mathbf{u}_1 = 0$ !!

Force: $\mathbf{F} = q\mathbf{E} \Rightarrow m_e \frac{\partial \mathbf{u}_1}{\partial t} = -e\mathbf{E}_1$

1st Maxwell: $\nabla \cdot \mathbf{E}_1 = -e n_1/\epsilon_0$

$$\Rightarrow \;\; \frac{\partial^2 n_1}{\partial t^2} + \left(\frac{n_0 e^2}{\epsilon_0 m_e}\right) n_1 = 0$$

$\omega_{pe}^2 = \frac{n_0 e^2}{\epsilon_0 m_e}$ plasma frequency

# Useful rules-of-thumb

Plasma frequency (angular frequency) $\omega_{pe}^2 = \frac{n_0 e^2}{\epsilon_0 m_e}$

$$\boxed{f_{pe}(\mathrm{Hz}) = \frac{\omega_{pe}}{2\pi} \approx 9,0 \cdot \sqrt{n(\mathrm{m}^{-3})}}$$

Debye length $\boxed{\lambda_D(\mathrm{m}) \approx 7,4\sqrt{T(\mathrm{eV})/n(\mathrm{cm}^{-3})}}$ Note the units ! (1 eV ≈ $1.16 \cdot 10^4$ K)

Gyromotion in the magnetic field

$\mathbf{F} = q_\alpha(\mathbf{v} \times \mathbf{B}) \Rightarrow$

$$\omega_{c\alpha} = \frac{|q_\alpha|B}{m_\alpha} \;\; ; \;\; f_{c\alpha} = \frac{\omega_{c\alpha}}{2\pi} \qquad \boxed{\begin{aligned} f_{ce}(\mathrm{Hz}) &\approx 28 \cdot B(\mathrm{nT}) \\ f_{cp}(\mathrm{Hz}) &\approx 1.5 \cdot 10^{-2} \cdot B(\mathrm{nT}) \end{aligned}}$$

## Plasma physics is difficult – but why?

- Combination of statistical physics and electromagnetism
- Large variety of scales, from electrons to ions to fluids
- A great variety of plasma descriptions must be mastered
  - single particle motion
  - Vlasov theory (electrons and ions described by distribution functions)
  - fluid descriptions (e.g., magnetohydrodynamics)
  - various hybrids of these
- Collisions or their absence

## Electrodynamics: Maxwell's equations

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{E} &= \rho/\epsilon_0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \end{aligned}}$$

$$\begin{aligned} c &= 1/\sqrt{\epsilon_0\mu_0} = 299792458 \text{ m/s} \\ \epsilon_0 &\approx 8,854 \cdot 10^{-12} \text{ As/Vm}, \\ \mu_0 &= 4\pi \cdot 10^{-7} \text{ Vs/Am}, \end{aligned}$$

Magnetic flux $\boxed{\Phi = \int_S \mathbf{B} \cdot d\mathbf{a}}$ is important in macroscopic plasma physics

EM fields are empirically determined through the Lorentz force

$\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$ or $\mathbf{F} = \frac{d}{dt}(\gamma m \mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

Because $\mathbf{v} \cdot \mathbf{F} = q(\mathbf{v} \cdot \mathbf{E} + \mathbf{v} \cdot \mathbf{v} \times \mathbf{B}) = q(\mathbf{v} \cdot \mathbf{E})$

only $\mathbf{E}$ performs work on charges $\frac{dW}{dt} = \frac{d}{dt}(\gamma m c^2) = q\mathbf{E} \cdot \mathbf{v}$

Thus any "magnetic acceleration" is associated with an electric field in the frame of reference where the acceleration is observed

# Ohm's law

Ohm's law $\boxed{\mathbf{J} = \sigma \cdot \mathbf{E}}$ relating the electric current and electric field is similar to the other constitutive equations $\mathbf{D} = \epsilon \cdot \mathbf{E}$ and $\mathbf{B} = \mu \cdot \mathbf{H}$

The conductivity $\sigma$, permittivity $\varepsilon$, and permeability $\mu$ depend on the electric and magnetic properties of the media considered. They may be scalars or tensors, and there does not need to be a local constitutive relation at all, not even Ohm's law!

A medium is called linear if $\varepsilon$, $\mu$, $\sigma$ are scalars and they are not functions of time and space.

Note that also in linear media $\varepsilon = \varepsilon(\omega, \mathbf{k})$, which is a very important relationship in plasma physics!

# Conservation of EM energy Poynting's theorem

The energy of electromagnetic field is given by

$$W_{EM} = \frac{1}{2}\int (\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\, d^3r \qquad w_E = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} \qquad w_M = \frac{1}{2}\mathbf{H}\cdot\mathbf{B}$$

Strating from Maxwell's equations it is straightforward to get

$$\boxed{\nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J} - \mathbf{E}\cdot\partial\mathbf{D}/\partial t - \mathbf{H}\cdot\partial\mathbf{B}/\partial t} \qquad \text{Poynting's theorem}$$

where $\boxed{\mathbf{S} = \mathbf{E}\times\mathbf{H}}$ is the Poynting vector

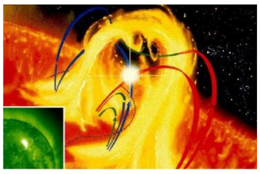
Integrating over volume $V$ (and using Gauss's law for the divergence)

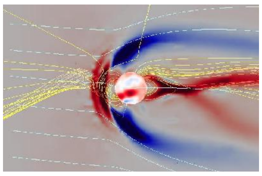
$$\underbrace{-\int_V \mathbf{J}\cdot\mathbf{E}\, d^3r}_{\text{work performed by the EM field}} = \underbrace{\oint_{\partial V} \mathbf{S}\cdot d\mathbf{a}}_{\text{energy flux through the surface of } V} + \underbrace{\int_V \frac{\partial}{\partial t}(w_E + w_M)\, d^3r}_{\text{change of energy in } V}$$

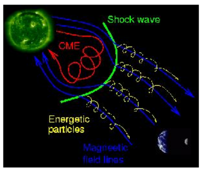
## Example: Poynting's theorem in fluid plasma (MHD)

$$-\oint_{\partial \mathcal{V}} \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a} = \frac{\partial}{\partial t} \int_{\mathcal{V}} \frac{B^2}{2\mu_0} d^3r + \int_{\mathcal{V}} \frac{J^2}{\sigma} d^3r + \int_{\mathcal{V}} \mathbf{V} \cdot \mathbf{J} \times \mathbf{B}\, d^3r$$

| EM energy flux into (out of) volume $\mathcal{V}$ | change of magnetic energy in volume $\mathcal{V}$ | plasma heating in volume $\mathcal{V}$ | acceleration in volume $\mathcal{V}$ |
|---|---|---|---|








# Single-particle motion: Guiding centre approximation

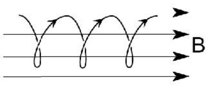
Equation of motion of charged particles is

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{F}_{non-EM}$$

(assume, for the time being, nonrelativistic motion; $\gamma = 1$ and $\mathbf{p} = m\mathbf{v}$)

Consider the case $\boxed{\mathbf{E} = 0 \text{ and } \mathbf{B} = \text{const}}$ (neglect the non-EM forces)

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B})$$




$$\boxed{\omega_c = \frac{qB}{m}}$$

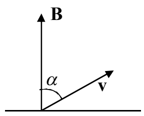
cyclotron frequency
gyro frequency
Larmor frequency

The radius of the circle is (Larmor radius) $r_L = \dfrac{v_\perp}{|\omega_c|} = \dfrac{mv_\perp}{|q|B}$ ; $v_\perp = \sqrt{v_x^2 + v_y^2}$

The gyro period (cyclotron period, Larmor time) is: $\tau_L = \dfrac{2\pi}{|\omega_c|}$

The pitch angle ($\alpha$) of the helical path is defined by

$$\tan\alpha = v_\perp / v_\parallel$$
$$\alpha = \arcsin(v_\perp / v) = \arccos(v_\parallel / v)$$



The frame of reference where $v_{\|} = 0$ : Guiding centre system (GCS)

Decomposition of the motion to the motion of the guiding centre and to the gyro motion is called the **guiding centre approximation**

In the GCS the charge causes an electric current: $I = q / \tau_L$

The magnetic moment associated to the circular loop is

$$\boxed{\mu = I \pi r_L^2 = \frac{1}{2}\frac{q^2 r_L^2 B}{m} = \frac{1}{2}\frac{m v_\perp^2}{B} = \frac{W_\perp}{B}}$$

or, in the vector form $\boxed{\boldsymbol{\mu} = \frac{1}{2} q\, \mathbf{r}_L \times \mathbf{v}_\perp}$

Clearly: $\boldsymbol{\mu}$ is always opposite to $\mathbf{B}$ ($\mathbf{r}_L$ depends on the sign of $q$)

Thus plasma can be considered a diamagnetic medium:

# E x B drift

Let $\mathbf{E} = \text{const}$ and $\mathbf{B} = \text{const}$

The eq. of motion along $\mathbf{B}$ is $m\dot{v}_{\|} = qE_{\|}$

→ constant acceleration parallel/antiparallel to **B**

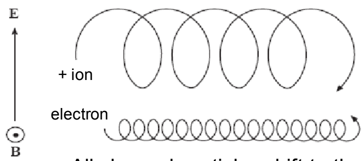
→ very rapid cancellation of large-scale $\mathbf{E}_{\|}$ in plasma!

The perpendicular components of the eq. of motion are

$$\begin{aligned} \dot{v}_x &= \omega_c v_y + \frac{q}{m} E_x \\ \dot{v}_y &= -\omega_c v_x \,. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \ddot{v}_x &= -\omega_c^2 v_x \\ \ddot{v}_y &= -\omega_c^2 \left( v_y + \frac{E_x}{B} \right) \end{aligned}$$

Substitution $v_y' = v_y + E_x/B$ leads again to gyro motion but now the GC drifts in the $y$-direction with speed $E_x/B$

In vector form: $\boxed{\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}}$




All charged particles drift to the same direction $\perp\mathbf{E}$ and $\perp\mathbf{B}$

# Other non-magnetic drifts

Write the perpendicular eq. of motion in the form $\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}(\mathbf{v}_\perp \times \mathbf{B}) + \frac{\mathbf{F}_\perp}{m}$

Assume that $\mathbf{F}_\perp$ gives rise to a drift $\mathbf{v}_D$ and transform $\mathbf{v}_\perp = \mathbf{v}'_\perp + \mathbf{v}_D$

$$\Rightarrow \frac{d\mathbf{v}'_\perp}{dt} = \frac{q}{m}(\mathbf{v}'_\perp \times \mathbf{B}) + \frac{q}{m}(\mathbf{v}_D \times \mathbf{B}) + \frac{\mathbf{F}_\perp}{m}$$

In GCS the last two terms must sum to $0 \Rightarrow$ $\boxed{\mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}}$ $(*)$

This requires $F/qB << c$. If $F > qcB$ , the GC approximation cannot be used!

Inserting $\mathbf{F}_\perp = q\mathbf{E}$ into $(*)$ we get the ExB-drift

$\mathbf{F}_\perp = m\mathbf{g}$ gives the gravitational drift $\boxed{\mathbf{v}_g = \frac{m\mathbf{g} \times \mathbf{B}}{qB^2} \propto \frac{m}{q}}$ separates charges $\rightarrow$ current

Slow time variations in $\mathbf{E} \rightarrow$ polarization drift $\boxed{\mathbf{v}_P = \frac{1}{\omega_c B}\frac{d\mathbf{E}_\perp}{dt}}$

The corresponding polarization current is

$$\mathbf{J}_P = n_e e(\mathbf{v}_{Pi} - \mathbf{v}_{Pe}) = \frac{n_e(m_i + m_e)}{B^2}\frac{d\mathbf{E}_\perp}{dt} \simeq \frac{n_e m_i}{B^2}\frac{d\mathbf{E}_\perp}{dt}$$

carried by ions!

# Magnetic drifts

Assume static but inhomogeneous magnetic field. Guiding centre approximation is useful if the spatial and temporal gradients of $\mathbf{B}$ are small as compared to the gyro motion:

$$|\partial \mathbf{B}/\partial t| \ll \omega_c B \;\; ; \;\; |\nabla B|_\perp \ll B/r_L \;\; ; \;\; |\nabla B|_\parallel \ll (\omega_c/v_\parallel)B$$

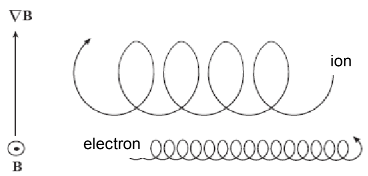
Considering first the gradient of $B$ only the force on the guiding centre can be shown to be $\mathbf{F} = -\mu\nabla B$

The parallel force gives acceleration along $\mathbf{B}$ $\boxed{\frac{d\mathbf{v}_\parallel}{dt} = -\frac{\mu}{m}\nabla_\parallel B}$

From the equation $\mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}$

we get the gradient drift velocity

$$\boxed{\mathbf{v}_G = \frac{\mu}{qB^2}\mathbf{B} \times (\nabla B) = \frac{W_\perp}{qB^3}\mathbf{B} \times (\nabla B)}$$



The field-line curvature (centrifugal force) leads to curvature drift

$$\boxed{\mathbf{v}_C = \frac{-mw_\parallel^2}{qB^2}\frac{\mathbf{R}_C \times \mathbf{B}}{R_C^2} = \frac{mw_\parallel^2}{qB^4}\mathbf{B} \times (\mathbf{B} \cdot \nabla)\mathbf{B}}$$

curvature radius

If there are on local currents in plasma, i.e., $\nabla \times \mathbf{B} = 0$

the curvature drift reduces to $\mathbf{v}_C = \frac{2W_\parallel}{qB^3}\mathbf{B} \times \nabla \mathbf{B}$

and we can combine the gradient and curvature drifts

$$\mathbf{v}_{GC} = \frac{W_\perp + 2W_\parallel}{qB^3}\mathbf{B} \times \nabla \mathbf{B} = \frac{W}{qBR_C}(1 + \cos^2 \alpha)\, \mathbf{n} \times \mathbf{t}$$

$\mathbf{t} \parallel \mathbf{B}$ & $\mathbf{n} \parallel \mathbf{R}_C$ are unit vectors

These are straightforward to modify for relativistic motion by substitution $\gamma m \rightarrow m$

# Adiabatic invariants

Symmetry principles: periodic motion $\leftrightarrow$ conserved quantity

symmetry $\leftrightarrow$ conservation law

What if the motion is almost periodic?

Hamiltonian mechanics:

Let $q$ & $p$ be canonical variables and the motion almost periodical $\Rightarrow$

$I = \oint p dq$ is constant, called adiabatic invariant

Example: Consider a charged particle in Larmor motion.
Assume that the $\mathbf{B}$ does not change much during within one circle.
The canonical coordinate is $\mathbf{r}_L$ and the canonical momentum $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$

charge!

$$\begin{aligned} I &= \oint \mathbf{p}_\perp \cdot d\mathbf{r}_L = \oint m\mathbf{v}_\perp \cdot d\mathbf{r}_L + q\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \\ &= \int_0^{2\pi r_L} mv_\perp dl + q\int_S \mathbf{B} \cdot d\mathbf{S} \\ &= 2\pi m v_\perp r_L - |q| B \pi r_L^2 = \frac{2\pi m}{|q|}\mu \end{aligned}$$

The magnetic moment is an adiabatic invariant

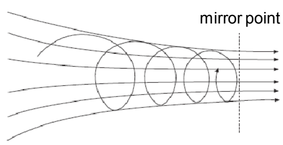
## Magnetic mirror

In guiding centre approximation both $W$ and $\mu$ are conserved.
If $B$ increases slowly, $W_\perp$ increases slowly, thus $W_\parallel$ decreases.
What happens when $W_\parallel \to 0$ ?

$$\mu = \frac{mv^2 \sin^2 \alpha}{2B}$$

As $\mu$ and $v^2$ are conserved, $\alpha$ and $B$ are related through

$$\frac{\sin^2 \alpha_1}{\sin^2 \alpha_2} = \frac{B_1}{B_2}$$




When $\alpha \to \pi/2$, the force $\mathbf{F} = -\mu \nabla_\parallel B$ on the GC turns the charge back (mirror force) and the mirror field $B_m$ for a charge that at $B_0$ has the pitch-angle $\alpha$ is given by $\sin^2 \alpha_0 = B_0/B_m$

Also parallel electric field and/or gravitation may need to be considered

$$m\frac{d\mathbf{v}_\parallel}{dt} = q\mathbf{E}_\parallel + m\mathbf{g}_\parallel - \mu \nabla_\parallel B$$

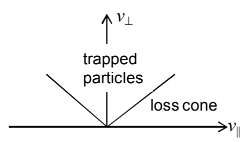
If the non-magnetic forces can be derived from a potential $U(s)$,

$$m\frac{dv_\parallel}{dt} = -\frac{\partial}{\partial s}[U(s) + \mu B(s)]$$

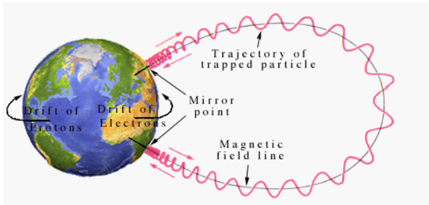
## Magnetic bottle

A simple magnetic bottle consists of two mirrors facing each other.
A charged particle is trapped in the bottle if

$$\arcsin\sqrt{\frac{B_0}{B_m}} \le \alpha_0 \le 180 - \arcsin\sqrt{\frac{B_0}{B_m}}$$




Otherwise it is said to be in the loss-cone and escape at the end of the bottle




The dipole field of the Earth is a large magnetic bottle

Note that there are much more complicated trapping schemes (e.g. tokamak)

In plasma physics $\mu$ is called the first adiabatic invariant.

Consider the bounce period of a charge in a magnetic bottle

$$\tau_b = 2\int_{s_m}^{s'_m} \frac{ds}{v_\parallel(s)} = \frac{2}{v}\int_{s_m}^{s'_m} \frac{ds}{(1 - B(s)/B_m)^{1/2}}$$

If $\tau_b \frac{dB/dt}{B} \ll 1$ then $J = \oint p_\parallel ds$ is constant (second adabatic invariant)

This, of course requires that $\tau_b \gg \tau_L$

If the perpendicular drift of the GC is nearly periodic (e.g. in a dipole field), the magnetic flux through the GC orbit

$\Phi = \oint \mathbf{A} \cdot d\mathbf{x}$ is conserved. This is the third adiabatic invariant.

The adiabatic invariants can be used as coordinates in studies of the evolution of the distribution function

$$f = f(\mu, J, \Phi)$$

# Betatron acceleration

Let $T$ be the kinetic energy of the particle in a time-dependent $\mathbf{B}$

Write the time derivative in a moving frame as $d/dt = \partial/\partial t + \mathbf{w} \cdot \nabla$

In GCS:

$$\frac{dT_{GCS}}{dt} = \mu\frac{dB}{dt} = \mu\left(\frac{\partial B}{\partial t} + \mathbf{w}_\perp \cdot \nabla_\perp B + w_\parallel \frac{\partial B}{\partial s}\right)$$

In the reference frame of the observer (OFR)

$$\frac{dT_{OFR}}{dt} = \frac{dT_{GCS}}{dt} + \frac{d}{dt}\left(\frac{1}{2}mw_\parallel^2\right) + \frac{d}{dt}\left(\frac{1}{2}mw_\perp^2\right)$$

Do a little algebra $\Rightarrow$ betatron acceleration:

$$\frac{dT_{OFR}}{dt} = \mu\frac{\partial B}{\partial t} + q\mathbf{w} \cdot \mathbf{E}$$

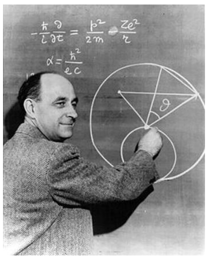
Increasing B at the position of GC: "gyro betatron"

Two effects:

- Field-aligned acceleration, if $E_\parallel \neq 0$
- Drift betatron:
  Particle drifts toward increasing $B$ : $B_2 > B_1$
  Conservation of $\mu$ $\Rightarrow$
  $\frac{W_{\perp 2}}{W_{\perp 1}} = \frac{B_2}{B_1}$ thus $W_{\perp 2} > W_{\perp 1}$

# Fermi acceleration of cosmic rays

Fermi proposed the drift betatron acceleration as a mechanism to accelerate cosmic rays to very large energies in 1949 in the following form:



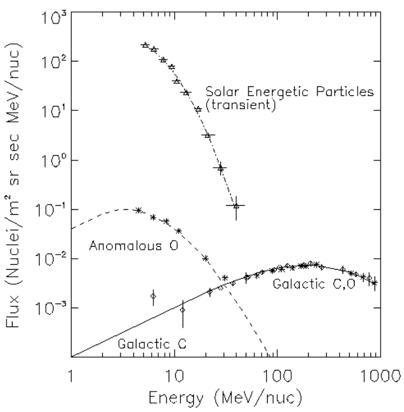
Let the particle move in a mirror field configuration where the mirror points move toward each other. Assume that $J$ is conserved.

Now $\oint ds$ decreases. To compensate this $v_{\|}$ and thus $W_{\|}$ must increase.
Compare this to the acceleration of a tennis ball hit by a racket!

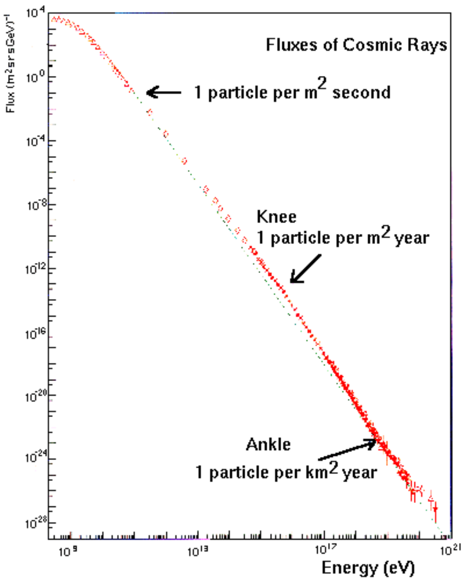
The modern version of Fermi acceleration is called diffusive shock acceleration where shock waves are responsible for the acceleration. It does not conserve $\mu$ and $J$.

## Cosmic Rays

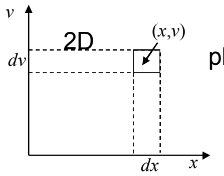
| | |
|---|---|
| galactic: | > 100 MeV |
| solar: | < 1 Gev |
| Anomalous: | around 10 MeV |



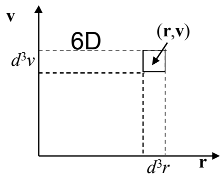

Spectrum of galactic cosmic rays



# Distribution function




phase space




A plasma particle ($i$) is at time $t$ in location $\mathbf{r}_i(t)$ and has velocity $\mathbf{v}_i(t)$

The distribution function $f(\mathbf{r}, \mathbf{v}, t)$ gives the particle number density in the ($\mathbf{r}$,$\mathbf{v}$) phase space element $dxdydzdv_xdv_ydv_z$ at time $t$

The units of $f$: volume$^{-1}$ x (volume of velocity space)$^{-1}$ = s$^3$m$^{-6}$

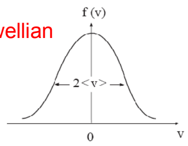
Normalization: $\int_V \int_{\mathbf{v}} f(\mathbf{r}, \mathbf{v}, t) d^3r d^3v = N \longleftarrow$ total number of particles

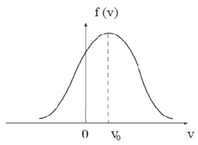
Average density: $\langle n \rangle = N/V$ ; density at location $\mathbf{r}$: $n(\mathbf{r}, t) = \int f(\mathbf{r}, \mathbf{v}, t) d^3v$

Example: Maxwellian distribution $f(v) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp\left( -\frac{mv^2}{2k_B T} \right)$, $n = \langle n \rangle$

# Examples of distribution functions

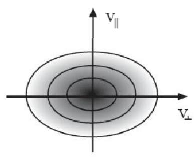
Maxwellian




$$f(v) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp\left( -\frac{m(\mathbf{v} - \mathbf{V}_0)^2}{2k_B T} \right)$$

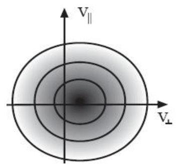
Maxwellian in a frame of reference that moves with velocity $\mathbf{V}_0$




Anisotropic (pancake) distribution ($\mathbf{v}_\parallel \parallel \mathbf{B}$)

$$f(v_\perp, v_\parallel) = \frac{n}{T_\perp T_\parallel^{1/2}} \left( \frac{m}{2\pi k_B} \right)^{3/2} \exp\left( -\frac{mv_\perp^2}{2k_B T_\perp} - \frac{mv_\parallel^2}{2k_B T_\parallel} \right)$$

Anisotropy can also be cigar-shaped (elongated in the direction of $\mathbf{B}$)



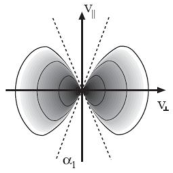

Drifting Maxwellian

$$f(v_\perp, v_\parallel) = \frac{n}{T_\perp T_\parallel^{1/2}} \left( \frac{m}{2\pi k_B} \right)^{3/2} \exp\left( -\frac{m(\mathbf{v}_\perp - \mathbf{v}_{0\perp})^2}{2k_B T_\perp} - \frac{mv_\parallel^2}{2k_B T_\parallel} \right)$$

Magnetic field-aligned beam (e.g., particles causing the aurora):

$$f(v_\perp, v_\parallel) = \frac{n}{T_\perp T_\parallel^{1/2}} \left(\frac{m}{2\pi k_B}\right)^{3/2} \exp\left(-\frac{m v_\perp^2}{2k_B T_\perp} - \frac{m(v_\parallel - v_{0\parallel})^2}{2k_B T_\parallel}\right)$$

Loss-cone distribution in a magnetic bottle:



Kappa distribution ~ Maxwellian with high-energy tail

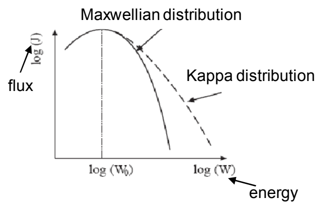
The tail follows a power law $f(W) \propto (W_0/W)^\kappa$

$$f_\kappa(W) = n\left(\frac{m}{2\pi\kappa W_0}\right)^{3/2} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)} \left(1 + \frac{W}{\kappa W_0}\right)^{-(\kappa+1)}$$

$[f_\kappa] = \mathrm{m}^{-6}\mathrm{s}^3$

$\Gamma$-function

energy at the peak of the distribution




Observed particle distributions often resemble kappa distributions;
a signature that non-thermal acceleration has taken place somewhere

# Vlasov and Boltzmann equations

equation(s) of motion for $f$

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0}$$

Vlasov equation (VE)

Compare with the Boltzmann equation in statistical physics (BE)

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_c$$

Boltzmann derived $(\partial f/\partial t)_c$ for strong short-range collisions

Ludwig Boltzmann

In plasmas most collisions are long-range small-angle collisions.
They are taken care by the average Lorentz force term

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_c}$$

large-angle collisions only
e.g., charge vs. neutral

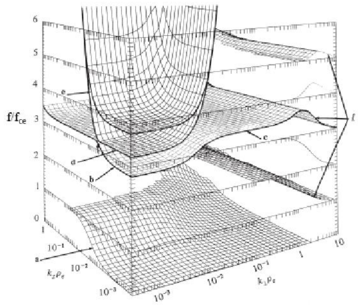
VE is often called collisionless Boltzmann equation

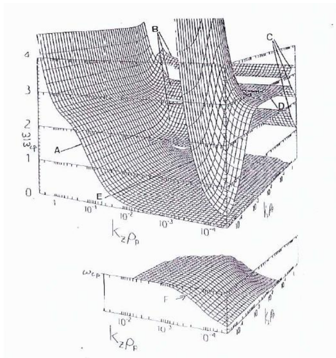
(M. Rosenbluth: actually a Bolzmann-less collision equation!)

## Vlasov theory

How to solve

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0}$$





## Landau's solution of VE

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0$$

Very hard task in a general case.
VE is nonlinear, thus we linearize:

$$\begin{aligned} f_\alpha &= f_{\alpha 0} + f_{\alpha 1} \\ \mathbf{E} &= \mathbf{E}_0 + \mathbf{E}_1 \\ \mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1 \end{aligned}$$

Consider a homogeneous, field-free plasma $\mathbf{E}_0 = \mathbf{B}_0 = 0$
in an electrostatic approximation: $\mathbf{E}_1 = -\nabla\varphi_1;\ \mathbf{B}_1 = 0$

The linearized VE is now $\frac{\partial f_{\alpha 1}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{\alpha 1}}{\partial \mathbf{r}} - \frac{q_\alpha}{m_\alpha}\frac{\partial \varphi_1}{\partial \mathbf{r}}\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0$

where $\nabla^2\varphi_1 = -\frac{1}{\epsilon_0}\sum_\alpha n_\alpha q_\alpha \int f_{\alpha 1}\, d^3v$

Vlasov tried this at the end of the 1930s using Fourier transformation

$\rightarrow$ an integral of the form $\int_{-\infty}^{\infty} \frac{\partial f_{\alpha 0}/\partial v}{\omega - kv}\, dv$

pole along the integration path, what to do?

In 1945 Vlasov presented a solution of VE at the long wavelength limit $(\omega \gg kv)$

$$\omega^2 \approx \omega_p^2 + \frac{3}{2}k^2 v_{th}^2 \qquad \text{Langmuir wave}$$

Thus in finite temperature the plasma oscillation propagates as a wave


Lev Landau

In 1946 Lev Landau found the way to handle the pole at $v = \omega/k$

He used the Fourier method in space but treated the problem as an initial value problem and used Laplace transform in time.

The solution for $E$ is: $E = E_0 \exp[-i(\omega + i\gamma)t] \;\propto\; \exp(\gamma t)$

For Maxwellian $f_0$ $\gamma < 0$ and the wave is damped: Landau damping

For an interested student: the long wavelength solution is

$$\begin{cases} \omega_r \approx \omega_{pe}(1+3k^2\lambda_{De}^2)^{1/2} \approx \omega_{pe}(1+\frac{3}{2}k^2\lambda_{De}^2) & \text{Langmuir wave} \\ \omega_i = -\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{|k^3\lambda_{De}^3|}\exp\left(-\frac{1}{2k^2\lambda_{De}^2} - \frac{3}{2}\right) & \text{Landau damping} \end{cases}$$

For details, see any advanced plasma physics text book

More realistic configurations soon become manually intractable, already for uniformly magnetized plasma the problem is to solve

$$|\mathcal{K}(\omega, \mathbf{k})| = 0$$

where

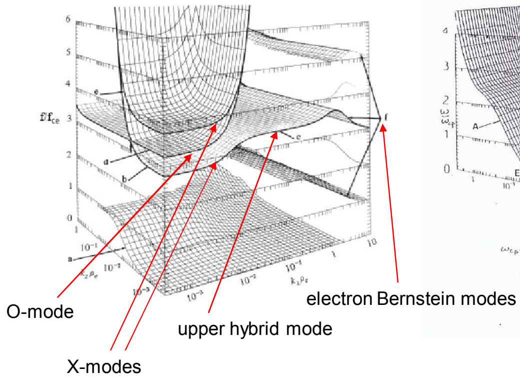
$$\mathcal{K}(\omega,\mathbf{k}) = \left(1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\right)\mathcal{I} - \sum_\alpha \sum_{n=-\infty}^{\infty} \frac{2\pi\omega_{p\alpha}^2}{n_{\alpha 0}\omega^2} \int_0^\infty \int_{-\infty}^{\infty} v_\perp dv_\perp dv_\parallel \left(k_\parallel \frac{\partial f_{\alpha 0}}{\partial v_\parallel} + \frac{n\omega_{c\alpha}}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp}\right) \frac{\mathcal{S}_{n\alpha}(v_\parallel, v_\perp)}{k_\parallel v_\parallel + n\omega_{c\alpha} - \omega}$$

$$\mathcal{S}_{n\alpha}(v_\parallel, v_\perp) = \begin{bmatrix} \frac{n^2\omega_{c\alpha}^2}{k_\perp^2}J_n^2 & \frac{inv_\perp\omega_{c\alpha}}{k_\perp}J_nJ_n' & \frac{nv_\parallel\omega_{c\alpha}}{k_\perp}J_n^2 \\ -\frac{inv_\perp\omega_{c\alpha}}{k_\perp}J_nJ_n' & v_\perp^2 J_n'^2 & -iv_\parallel v_\perp J_n J_n' \\ \frac{nv_\parallel\omega_{c\alpha}}{k_\perp}J_n^2 & iv_\parallel v_\perp J_nJ_n' & v_\parallel^2 J_n^2 \end{bmatrix}$$

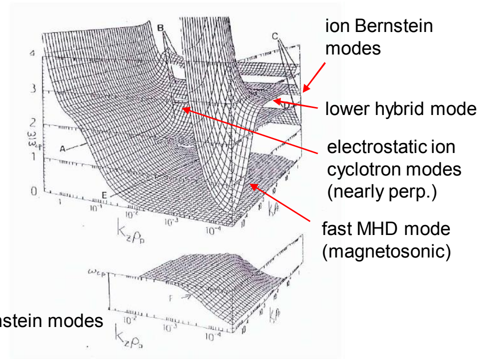
$$J_n' = dJ_n/d(k_\perp v_\perp/\omega_{c\alpha})$$

# Perpendicular modes on dispersion surfaces

"electron modes"

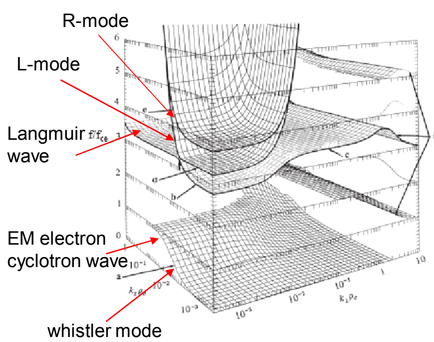



"ion modes"

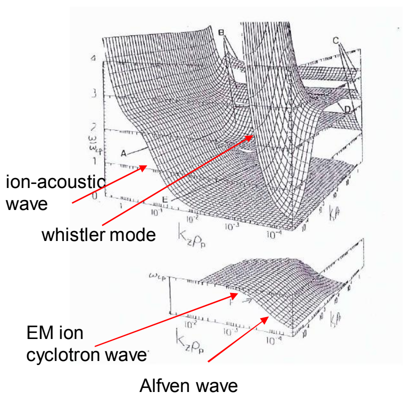



# Parallel modes on dispersion surfaces

"electron modes"




"ion modes"



## Macroscopic theory: Velocity moments of $f$

$$\int f\ d^3v\,;\quad \int \mathbf{v}f\ d^3v\,;\quad \int \mathbf{v}\mathbf{v}f\ d^3v$$

$$n(\mathbf{r},t) = \int f(\mathbf{r},\mathbf{v},t)d^3v$$ Density is the zeroth moment; $[n] = \mathrm{m}^{-3}$

The first moment ($\alpha$ denotes different particle species):

$$\mathbf{\Gamma}_\alpha(\mathbf{r},t) = \int \mathbf{v}f_\alpha(\mathbf{r},\mathbf{v},t)d^3v$$ Particle flux; $[\Gamma] = \mathrm{m}^{-2}\ \mathrm{s}^{-1}$

$$\mathbf{V}_\alpha(\mathbf{r},t) = \frac{\int \mathbf{v}f_\alpha(\mathbf{r},\mathbf{v},t)d^3v}{\int f_\alpha(\mathbf{r},\mathbf{v},t)d^3v}$$ Average velocity = flux/density, $[V] = \mathrm{m\ s}^{-1}$

DO NOT EVER MIX UP $\mathbf{V}(\mathbf{r},t)$ and $\mathbf{v}(t)$ !!

$$\mathbf{J}_\alpha(\mathbf{r},t) = q_\alpha\mathbf{\Gamma}_\alpha = q_\alpha n_\alpha \mathbf{V}_\alpha$$ Electric current density, $[J] = \mathrm{C\ m}^{-2}\ \mathrm{s}^{-1} = \mathrm{A\ m}^{-2}$

## Pressure and temperature

from the second velocity moments

Pressure tensor $$\mathcal{P}_\alpha(\mathbf{r},t) = m_\alpha \int \underbrace{(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)}_{\text{dyadic product } \rightarrow \text{ tensor}} f_\alpha(\mathbf{r},\mathbf{v},t)d^3v$$

If $\mathcal{P}_\alpha = p_\alpha \mathcal{I}$ where $\mathcal{I}$ is the unit tensor, we find the scalar pressure

$$p_\alpha = \frac{m_\alpha}{3}\int (\mathbf{v}-\mathbf{V}_\alpha)^2 f_\alpha(\mathbf{r},\mathbf{v},t)d^3v = n_\alpha k_B T_\alpha$$ introducing the temperature

Assume $\mathbf{V} = 0$: $$\frac{3}{2}k_B T_\alpha(\mathbf{r},t) = \frac{m_\alpha}{2}\frac{\int v^2 f_\alpha(\mathbf{r},\mathbf{v},t)d^3v}{\int f_\alpha(\mathbf{r},\mathbf{v},t)d^3v}$$ $\boxed{T \propto \langle \text{K.E.} \rangle}$

Thus we can calculate a "temperature" also in non-Maxwellian plasma!

Magnetic pressure
(i.e. magnetic energy density) $B^2/2\mu_0$

Plasma beta $$\beta = \frac{2\mu_0 \sum_\alpha n_\alpha k_B T_\alpha}{B^2}$$
thermal pressure / magnetic pressure

$\beta \ll 1$ $\mathbf{B}$ dominates over plasma
$\beta \gg 1$ plasma dominates over $\mathbf{B}$

3rd velocity moment → heat flux (temperature x velocity), etc. to higher orders…

# Macroscopic plasma description

Macroscopic plasma theories are fluid theories at different levels

- single fluid (magnetohydrodynamics MHD)
- two-fluid (multifluid, separate equations for electron and ion fluids)
- hybrid (fluid electrons with (quasi)particle ions)

Macroscopic equations can be obtained by taking velocity moments of Boltzmann / Vlasov equations

$$\int \mathbf{v}^n \left( \frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} \right) d^3v = \int \left( \frac{\partial f_\alpha}{\partial t} \right)_c d^3v$$

order $n$ order $n+1$

Taking the $n^{\text{th}}$ moment of BE/VE introduces terms of order $n+1$ !
This leads to an open chain of equations that must be terminated by applying some form of physical intution.

Note that the collision integrals can be very tricky!

We start from the Boltzmann equation

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \left( \frac{\partial f_\alpha}{\partial t} \right)_c$$

and calculate its zeroth velocity moment.

In absence of ionizing or recombining collisions, the collision integral is zero, and the result is the continuity equation

$$\boxed{\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{V}_\alpha) = 0}$$

Multiplying by mass or charge we get the continuity eqs for these

$$\frac{\partial \rho_{m\alpha}}{\partial t} + \nabla \cdot (\rho_{m\alpha} \mathbf{V}_\alpha) = 0 \qquad \frac{\partial \rho_{q\alpha}}{\partial t} + \nabla \cdot \mathbf{J}_\alpha = 0$$

General form of conservation law for $F$ :
($\mathbf{G}$ is the flux of $F$)

$$\boxed{\frac{\partial F}{\partial t} + \nabla \cdot \mathbf{G} = 0}$$

To calculate the first moment, multiply BE by momentum $m_\alpha \mathbf{v}$ and integrate

$$\Rightarrow \quad n_\alpha m_\alpha \frac{\partial \mathbf{V}_\alpha}{\partial t} + n_\alpha m_\alpha \mathbf{V}_\alpha \cdot \nabla \mathbf{V}_\alpha - n_\alpha q_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B}) + \nabla \cdot \mathcal{P}_\alpha = m_\alpha \int \mathbf{v} \left( \frac{\partial f_\alpha}{\partial t} \right)_c d^3v \,.$$

Equation for momentum transport, actually equation of motion!

$$n_\alpha m_\alpha \frac{\partial \mathbf{V}_\alpha}{\partial t} + n_\alpha m_\alpha \mathbf{V}_\alpha \cdot \nabla \mathbf{V}_\alpha - n_\alpha q_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B}) + \nabla \cdot \mathcal{P}_\alpha = m_\alpha \int \mathbf{v} \left( \frac{\partial f_\alpha}{\partial t} \right)_c d^3 v \,.$$

Now the convective derivative of $\mathbf{V}$ $\mathbf{V}_\alpha \cdot \nabla \mathbf{V}_\alpha$
and the pressure tensor $\mathcal{P}_\alpha$ are second moments

The electric and magnetic fields must fulfill Maxwell's equations
$(\rho_{ext}, \mathbf{J}_{ext})$ are external sources

$$\nabla \cdot \mathbf{E} = \sum_\alpha \frac{n_\alpha q_\alpha}{\epsilon_0} + \frac{\rho_{ext}}{\epsilon_0}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \sum_\alpha n_\alpha q_\alpha \mathbf{V}_\alpha + \mu_0 \mathbf{J}_{ext}$$

Note that the collision integral can be non-zero, because collisions transfer momentum between different particle species!

Calculate the second moment (multiply by $\mathbf{vv}$, and integrate; rather tedious)
→ heat transfer equation (conservation of energy)

Now the heat flux is of thrid order. To close the chain some equation relating the variables must be introduced.

# Equations of MHD

## Sum over all particle species

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0$$

$$\rho_m \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$ (isotropic pressure assumed)

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{J}/\sigma$$

$$p = p_0 \left( \frac{n}{n_0} \right)^\gamma$$ (or energy equation)

Relevant Maxwell's equations; displacement current neglected

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$ ($\Rightarrow$ $\nabla \cdot \mathbf{J} = 0$ )

In space plasmas the conductivity often is very large: ideal MHD $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$

However, sometimes other terms than the resistive start to play a role (e.g., in magnetic reconnection) and a more general Ohm's law is needed

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \frac{\mathbf{J}}{\sigma} + \frac{1}{ne} \mathbf{J} \times \mathbf{B} - \frac{1}{ne} \nabla \cdot \mathcal{P}_e + \frac{m_e}{ne^2} \frac{\partial \mathbf{J}}{\partial t}$$

## Convection and diffusion of B

Take curl of the MHD Ohm's law $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{J}/\sigma$ and apply Faraday's law

$$\Rightarrow \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B} - \mathbf{J}/\sigma)$$

Thereafter use Ampère's law and the divergence of $\mathbf{B}$ to get the induction equation for the magnetic field

$$\begin{cases} \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$$

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}}$$

(Note that $\sigma$ has been assumed constant here)

Assume that plasma does not move $(\mathbf{V} = 0)$

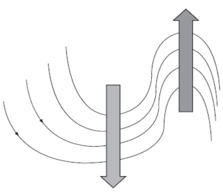
$\Rightarrow$ diffusion equation: $\boxed{\frac{\partial \mathbf{B}}{\partial t} = D_m \nabla^2 \mathbf{B}}$ diffusion coefficient: $D_m = (\mu_0 \sigma)^{-1}$

If the resistivity is finite, the magnetic field diffuses into the plasma to remove local magnetic inhomogeneities, e.g., curves in the field, etc.

Let $L_B$ the characteristic scale of magnetic inhomogeneities. The solution is $B = B_0 \exp(\pm t/\tau_d)$ where the characteristic diffusion time is $\boxed{\tau_d = \mu_0 \sigma L_B^2}$

In case of $\sigma \to \infty$ the diffusion becomes very slow and the evolution of B is completely determined by the plasma flow (field is frozen-in to the plasma)

$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})}$ convection equation



Let the characteristic spatial and temporal scales be $\nabla \to L_B^{-1}$ & $\partial/\partial t \to \tau^{-1}$ and the diffusion time $\tau_d = \mu_0 \sigma L_B^2$

The order of magnitude estimates for the terms of the induction equation are

$$\boxed{\frac{B}{\tau} = \frac{VB}{L_B} + \frac{B}{\tau_d}}$$

The measure of the relative strengths of convection and diffusion is the magnetic Reynolds number $R_m = \mu_0 \sigma L_B V = L_B V / D_m$

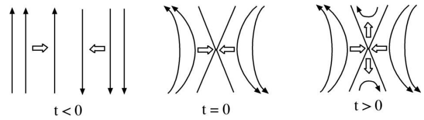
This is analogous to the Reynolds number in hydrodynamics $R = LV/\nu$

viscosity

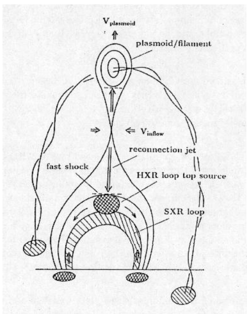
In fully ionized plasmas $R_m$ is often very large. E.g. in the solar wind at 1 AU it is $10^{16} - 10^{17}$. This means that during the 150 million km travel from the Sun the field diffuses about 1 km! Very ideal MHD: $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$

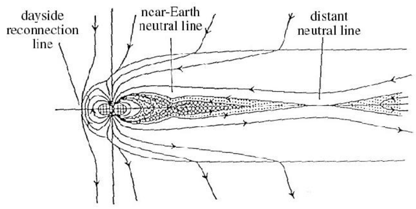
## Break-down of the frozen-in condition: Magnetic reconnection

Change of magnetic connection between two ideal MHD domains



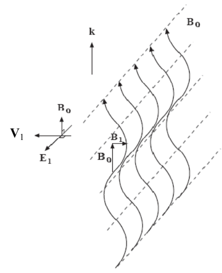

Large $R_m = \mu_0 \sigma V L$ allows formation of thin current sheets, e.g., solar atmosphere, magnetopause, tail current sheet




Magnetic reconnection is a fundamental energy release process in magnetized plasmas but we skip the details on this lecture

## Magnetohydrodynamic waves Alfvén waves




MHD is a fluid theory and there are similar wave modes as in ordinary fluid theory (hydrodynamics). In hydrodynamics the restoring forces for perturbations are the pressure gradient and gravity. Also in MHD the pressure force leads to acoustic fluctuations, whereas Ampère's force (**J**x**B**) leads to an entirely new class of wave modes, called Alfvén (or MHD) waves.

As the displacement current $c^{-2}\partial \mathbf{E}/\partial t$ is neglected in MHD, there are no electromagnetic waves of classical electrodynamics. Of course EM waves can propagate through MHD plasma (e.g. light, radio waves, etc.) and even interact with the plasma particles, but that is beyond the MHD approximation.

# Dispersion equation for ideal MHD waves

$$
\begin{aligned}
\frac{\partial \rho_m}{\partial t} + \nabla\cdot(\rho_m \mathbf{V}) &= 0 \\
\rho_m \frac{\partial \mathbf{V}}{\partial t} + \rho_m(\mathbf{V}\cdot\nabla)\mathbf{V} &= -\nabla p + \mathbf{J}\times\mathbf{B} \\
\nabla p &= v_s^2 \nabla \rho_m \\
\nabla\times\mathbf{B} &= \mu_0 \mathbf{J} \\
\nabla\times\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\mathbf{E} + \mathbf{V}\times\mathbf{B} &= 0\,.
\end{aligned}
$$

eliminate $\mathbf{J}$

eliminate $\nabla p$

eliminate $\mathbf{E}$

$$
\Rightarrow \quad
\begin{aligned}
\rho_m \frac{\partial \mathbf{V}}{\partial t} + \rho_m(\mathbf{V}\cdot\nabla)\mathbf{V} &= -v_s^2 \nabla \rho_m + (\nabla\times\mathbf{B})\times\mathbf{B}/\mu_0 \\
\nabla\times(\mathbf{V}\times\mathbf{B}) &= \frac{\partial \mathbf{B}}{\partial t}\,.
\end{aligned}
$$

We are left with 7 scalar equations for 7 unknowns ($\rho_{m0}$, $\mathbf{V}$, $\mathbf{B}$)

Consider small perturbations

$$
\begin{aligned}
\mathbf{B}(\mathbf{r},t) &= \mathbf{B}_0 + \mathbf{B}_1(\mathbf{r},t) \\
\rho_m(\mathbf{r},t) &= \rho_{m0} + \rho_{m1}(\mathbf{r},t) \\
\mathbf{V}(\mathbf{r},t) &= \mathbf{V}_1(\mathbf{r},t)\,.
\end{aligned}
$$

and linearize

$$
\Rightarrow \quad
\begin{aligned}
\frac{\partial \rho_{m1}}{\partial t} + \rho_{m0}(\nabla\cdot\mathbf{V}_1) &= 0 \qquad (*) \\
\rho_{m0}\frac{\partial \mathbf{V}_1}{\partial t} + v_s^2\nabla\rho_{m1} + \mathbf{B}_0\times(\nabla\times\mathbf{B}_1)/\mu_0 &= 0 \qquad (**) \\
\frac{\partial \mathbf{B}_1}{\partial t} - \nabla\times(\mathbf{V}_1\times\mathbf{B}_0) &= 0 \qquad (***)
\end{aligned}
$$

Find an equation for $\mathbf{V}_1$. Start by taking the time derivative of (**)

$$
\rho_{m0}\frac{\partial^2 \mathbf{V}_1}{\partial t^2} + v_s^2 \nabla\left(\frac{\partial \rho_{m1}}{\partial t}\right) + \frac{\mathbf{B}_0}{\mu_0}\times\left(\nabla\times\frac{\partial \mathbf{B}_1}{\partial t}\right) = 0
$$

Insert (*) and (***) and introduce the Alfvén velocity as a vector $\boxed{\mathbf{v}_A = \frac{\mathbf{B}_0}{\sqrt{\mu_0 \rho_{m0}}}}$

$$
\Rightarrow \quad \frac{\partial^2 \mathbf{V}_1}{\partial t^2} - v_s^2\nabla(\nabla\cdot\mathbf{V}_1) + \mathbf{v}_A\times\{\nabla\times[\nabla\times(\mathbf{V}_1\times\mathbf{v}_A)]\} = 0
$$

Look for plane wave solutions $\mathbf{V}_1(\mathbf{r},t) = \mathbf{V}_1 \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$

$$
\Rightarrow \quad -\omega^2\mathbf{V}_1 + v_s^2(\mathbf{k}\cdot\mathbf{V}_1)\mathbf{k} - \mathbf{v}_A\times\{\mathbf{k}\times[\mathbf{k}\times(\mathbf{V}_1\times\mathbf{v}_A)]\} = 0
$$

Using $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ a few times we have the dispersion equation for the waves in ideal MHD

$$
\boxed{
\begin{gathered}
-\omega^2\mathbf{V}_1 + (v_s^2 + v_A^2)(\mathbf{k}\cdot\mathbf{V}_1)\mathbf{k} \\
+(\mathbf{k}\cdot\mathbf{v}_A)[(\mathbf{k}\cdot\mathbf{v}_A)\mathbf{V}_1 - (\mathbf{v}_A\cdot\mathbf{V}_1)\mathbf{k} - (\mathbf{k}\cdot\mathbf{V}_1)\mathbf{v}_A] = 0
\end{gathered}
}
$$

**Propagation perpendicular to the magnetic field:** $\mathrm{k} \perp \mathrm{B}_0$

Now $\mathbf{k} \cdot \mathbf{v}_A = 0$ and the dispersion equation reduces to

$$-\omega^2 \mathbf{V}_1 + (v_s^2 + v_A^2)(\mathbf{k} \cdot \mathbf{V}_1)\mathbf{k} = 0$$

$$\Rightarrow \quad \mathbf{V}_1 = (v_s^2 + v_A^2)(\mathbf{k} \cdot \mathbf{V}_1)\mathbf{k}/\omega^2$$

clearly $\mathbf{k} \parallel \mathbf{V}_1$

And we have found the magnetosonic wave $\boxed{\omega/k = \sqrt{v_s^2 + v_A^2}}$

This mode has many names in the literature:
compressional Alfvén wave, fast Alfvén wave, fast MHD wave

**Propagation parallel to the magnetic field:** $\mathrm{k} \parallel \mathrm{B}_0$

$$(k^2 v_A^2 - \omega^2)\mathbf{V}_1 + \left(\frac{v_s^2}{v_A^2} - 1\right) k^2 (\mathbf{V}_1 \cdot \mathbf{v}_A)\mathbf{v}_A = 0$$

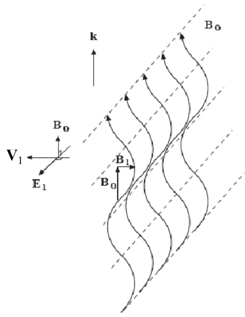
Two different solutions (modes)

1) $\mathbf{V}_1 \parallel \mathbf{B}_0 \parallel \mathbf{k} \;\Rightarrow\; \omega/k = v_s$ the sound wave

2) $\mathbf{V}_1 \perp \mathbf{B}_0 \parallel \mathbf{k} \;\Rightarrow\; \mathbf{V}_1 \cdot \mathbf{v}_A = 0$

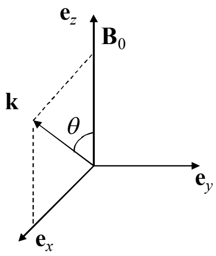
$$\Rightarrow \quad \omega/k = v_A$$

This mode is called Alfvén wave or shear Alfvén wave




**Propagation at an arbitrary angle**

$$\begin{aligned}
\mathbf{k} &= k(\mathbf{e}_x \sin\theta + \mathbf{e}_z \cos\theta) \\
\mathbf{v}_A &= v_A \mathbf{e}_z \\
\mathbf{V}_1 &= V_{1x}\mathbf{e}_x + V_{1y}\mathbf{e}_y + V_{1z}\mathbf{e}_z \\
\mathbf{k} \cdot \mathbf{v}_A &= k v_A \cos\theta \\
\mathbf{k} \cdot \mathbf{V}_1 &= k(V_{1x}\sin\theta + V_{1z}\cos\theta) \\
\mathbf{v}_A \cdot \mathbf{V}_1 &= v_A V_{1z}\,.
\end{aligned}$$




Dispersion equation $\Rightarrow$

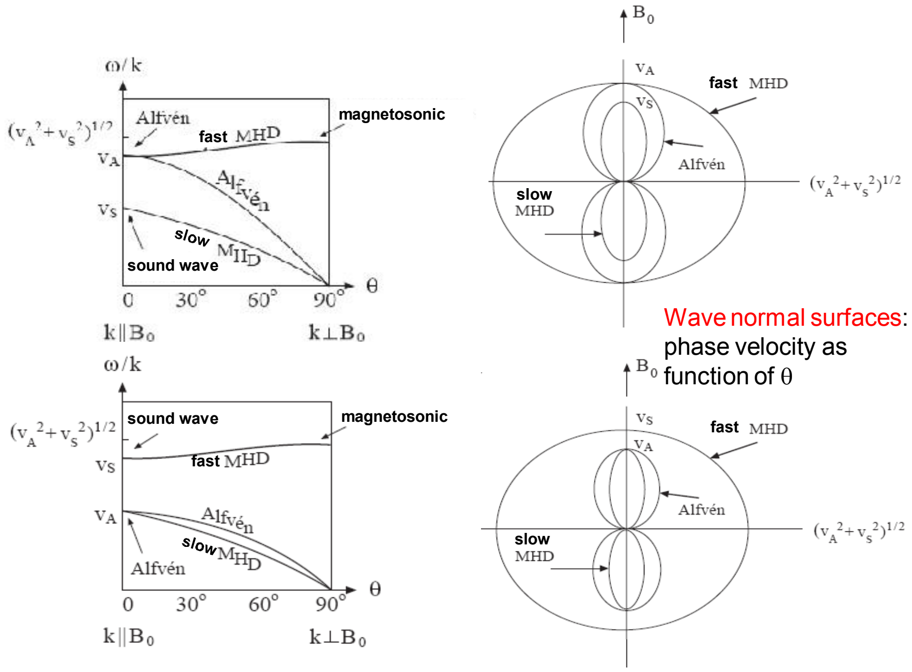
$$\begin{aligned}
V_{1x}(-\omega^2 + k^2 v_A^2 + k^2 v_s^2 \sin^2\theta) + V_{1z}(k^2 v_s^2 \sin\theta\cos\theta) &= 0 \\
V_{1y}(-\omega^2 + k^2 v_A^2 \cos^2\theta) &= 0 \\
V_{1x}(k^2 v_s^2 \sin\theta\cos\theta) + V_{1z}(-\omega^2 + k^2 v_s^2 \cos^2\theta) &= 0
\end{aligned}$$

Coeff. of $V_{1y}$ $\Rightarrow$ $\boxed{\omega/k = v_A\cos\theta}$ shear Alfvén wave

From the determinant of the remaining equations:

$$\boxed{\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(v_s^2 + v_A^2) \pm \frac{1}{2}[(v_s^2 + v_A^2)^2 - 4v_s^2 v_A^2 \cos^2\theta]^{1/2}}$$

Fast (+) and slow (–) Alfvén/MHD waves




Wave normal surfaces: phase velocity as function of θ

# Some remarks

- Collective effects of free charges determine the behavior of the plasma as an electromagnetic medium
- Plasma physics also relies on tools of statistical physics
- Plasma behaves nonlinearly
  - Vlasov equation is nonlinear, magnetohydrostatic equilibrium is nonlinear, etc.
  - linearizations are often useful, e.g., to find the normal modes of plasma oscillations, but the observable plasma oscillations are either damped or grow to nonlinear level leading to instabilities
- Plasma is often turbulent
  - plasma turbulence is an even more complicated issue than ordinary fluid turbulence
- Plasma systems often exhibit chaotic behavior
  - concepts of chaos, such as self-organized criticality, intermittence, renormalization groups, etc., are important in theoretical plasma physics.