Incompressible MHD simulations

Felix Spanier [1]
Lehrstuhl für Astronomie
Universität Würzburg

Simulation methods in astrophysics

Background

The dynamics of an incompressible fluid may be described by the Navier-Stokes-equation

$$\partial_t \boldsymbol{u} + \nabla \cdot (\boldsymbol{u} : \boldsymbol{u}) = \nu \Delta \boldsymbol{u} - \nabla p \tag{1}$$

where the solution $\boldsymbol{u}$ has to fulfill

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

This may be solved by using a classical solution of the Navier-Stokes-equation and projecting this solution on its solenoidal part or by making use of the pseudospectral method.
We are going for the latter solution. The equation is Fourier transformed to derive an ODE for $\tilde{\boldsymbol{u}}$.

Objective 1

Convince yourself that the pressure may be derived from the incompressible Navier-Stokes-equation , yielding

$$p = \frac{k_\alpha k_\beta}{k^2} (\widetilde{u^\alpha u^\beta}) \tag{3}$$

and that the incompressible Navier-Stokes-equation has the form

$$\frac{\partial \tilde{u}_\alpha}{\partial t} = -ik_\gamma \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) (\widetilde{u_\beta u_\gamma}) - \nu k^2 \tilde{u}_\alpha \tag{4}$$

$$\tag{5}$$

Objective 2

Write a two-dimensional pseudospectral code.
Consider the following

- $u^\alpha u^\beta$ has to be evaluated in real space
- The k are physical quantities. In numerical Fourier transforms k has values ranging from $0 \ldots N-1$ which represent physical k values $0, 1 \cdots -2, -1$
- After every Fourier transform the upper half of the k space is flawed by aliasing. Set that part to zero

Objective 3

Use

$$\boldsymbol{\Psi}(x,y) = \boldsymbol{e}_z \Big(\exp(-(x - \frac{1}{2}L - \Delta)^2 - (y - \frac{1}{2}L - \Delta)^2) \tag{6}$$

$$\pm \exp(-(x - \frac{1}{2}L + \Delta)^2 - (y - \frac{1}{2}L + \Delta)^2)\Big) \tag{7}$$

as initial data ($\boldsymbol{u} = \nabla \times \boldsymbol{\Psi}$) with a simulation box size $L \times L$.
Plot colorful pictures. Movies give extra points.