Incompressible MHD simulations

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Simulation methods in astrophysics

Outline

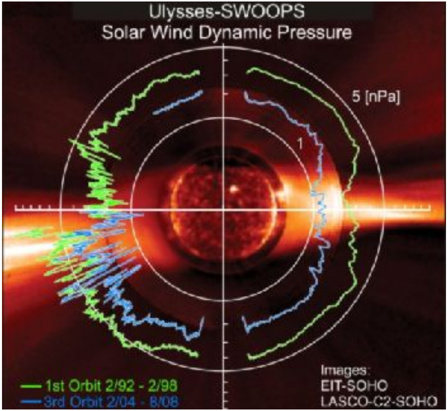
- Where do we need incompressible MHD?
- Theory of HD&MHD
- Turbulence
- Numerical methods for HD
- Numerical methods for MHD

Heliosphere

Closest available space plasma:
The solar wind

- includes shocks
- and density fluctuation

The large scale evolution seems to be compressible...

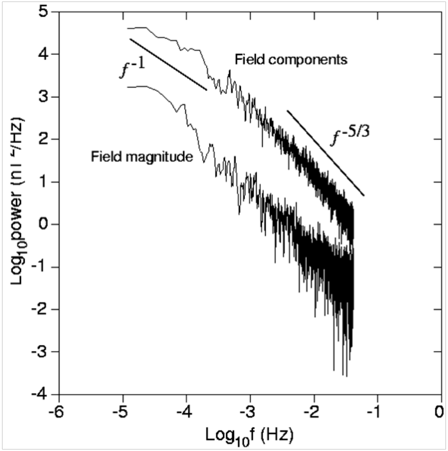


Heliosphere

. . . but in the frame of the solar wind, we find

- a turbulent spectrum
- Alfvén waves travelling

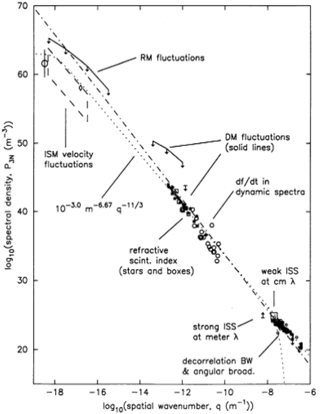
Sounds like incompressible turbulence!



Interstellar medium

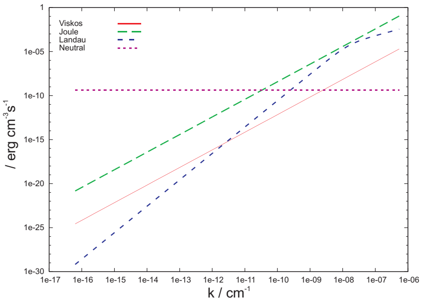
Infamous picture

- this is usually used to suggest turbulence
- it really shows density fluctuations
- Density fluctuations? That's compressible turbulence?
- Really?

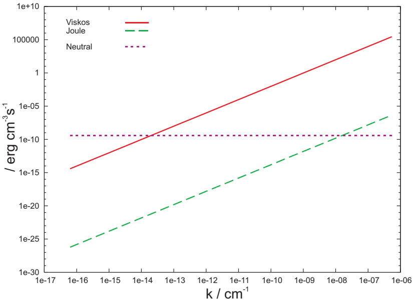


Interstellar medium

Damping rates



Alfvén wave damping



Sound wave damping

Interstellar medium

So is this incompressible MHD?

- Wave damping suggests that for large k only Alfvén waves survive
- Density fluctuations suggest there is compressible turbulence
- Two solutions:
- Alfvén turbulence with enslaved density fluctuations
- Not waves but shocks govern the ISM

Euler Equation

Force on a volume element of fluid

$$\boldsymbol{F} = -\oint p \, d\boldsymbol{f} \tag{1}$$

Divergence theorem

$$\boldsymbol{F} = -\int \nabla p \, dV \tag{2}$$

Newton's law for one volume element

$$m\frac{d\boldsymbol{u}}{dt} = \boldsymbol{F} \tag{3}$$

Using density instead of total mass

$$\int \rho \frac{d\boldsymbol{u}}{dt} \, dV = -\int \nabla p \, dV \tag{4}$$

$$\rho \frac{d\boldsymbol{u}}{dt} = -\nabla p \tag{5}$$

$\Rightarrow$ Lagrange picture

Euler Equation

Changing to material derivative

$$d\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial t} dt + (d\boldsymbol{r}\nabla)\boldsymbol{u} \tag{1}$$

$$\Rightarrow \frac{d\boldsymbol{u}}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\nabla)\boldsymbol{u} \tag{2}$$

Euler's equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p \tag{3}$$

Here the velocity is a function of space and time

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}, t) \tag{4}$$

This can now be applied to the momentum transport equation

$$\frac{\partial}{\partial t}(\rho u_i) = \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} \tag{5}$$

Euler Equation

Partial derivative of ***u*** is known from Euler

$$\frac{\partial u_i}{\partial t} = -u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \quad (1)$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho u_i) = -\rho u_k \frac{\partial u_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - u_i \frac{\partial \rho u_k}{\partial x_k} \quad (2)$$

$$= -\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}(\rho u_i u_k) \quad (3)$$

$$= -\delta_{ik}\frac{\partial p}{\partial x_k} - \frac{\partial}{\partial x_k}(\rho u_i u_k) \quad (4)$$

$$\frac{\partial \rho u_i}{\partial t} = -\frac{\partial}{\partial x_k}\Pi_{ik} \quad (5)$$

$$\Pi_{ik} = -p\delta_{ik} - \rho u_i u_k \quad (6)$$

Π_{ik} = stress tensor

Continuity equation

$$\frac{\partial}{\partial t}\int \rho \, dV = -\oint \rho \boldsymbol{u} \, dV \quad (7)$$

$$\int \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{u})\right) dV = 0 \quad (8)$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{u}) = 0 \quad (9)$$

For incompressible fluids (ρ=const)

$$\nabla\cdot \boldsymbol{u} = 0 \quad (10)$$

Viscid fluid

$$\Pi_{ik} = p\delta_{ik} + \rho u_i u_k - \sigma_{ik} \tag{11}$$

σ_{ik} = shear stress
For Newtonian fluids this depends on derivatives of the velocity

$$\sigma_{ik} = a\frac{\partial u_i}{\partial x_k} + b\frac{\partial u_k}{\partial x_i} + c\frac{\partial u_l}{\partial x_l}\delta_{ik} \tag{12}$$

Coefficients have to fulfill

- no viscosity in uniform fluids($\boldsymbol{u}$=const)
- no viscosity in uniform rotating floew

It follows $a = b$

$$\sigma_{ik} = \eta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_l}{\partial x_l}\right) + \zeta\delta_{ik}\frac{\partial u_l}{\partial x_l} \tag{13}$$

Viscid fluid

Stress tensor for incompressible fluids

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{11}$$

$$\Rightarrow \sigma_{ik} = \eta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{12}$$

$$\frac{\partial \sigma_{ik}}{\partial x_k} = \eta\frac{\partial^2 u_i}{\partial x_k^2} \tag{13}$$

And we find the *Navier-Stokes-equation* :

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\nabla)\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u} \tag{14}$$

$\nu = \frac{\eta}{\rho}$ = Viscosity

Pressure

$$\Delta p = -(\boldsymbol{u}\nabla)\boldsymbol{u} + \nu\Delta\boldsymbol{u} \tag{15}$$

Vorticity

Using $\nabla \cdot \boldsymbol{u} = 0$ one can take the curl of the Navier-Stokes-equation , resulting in a PDE for $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$

$$\frac{\partial}{\partial t}(\nabla \times \boldsymbol{u}) - \nabla \times (\boldsymbol{u} \times (\nabla \times \boldsymbol{u})) = -\nu \nabla \times (\nabla \times (\nabla \times \boldsymbol{u})) \quad (16)$$

$$\Rightarrow \frac{\partial \boldsymbol{\omega}}{\partial t} = (\omega \nabla)\boldsymbol{u} + \nu \Delta \boldsymbol{\omega} \quad (17)$$

ω is quasi-scalar for 2d-velocities
We need the stream function Ψ to derive velocities

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \quad (18)$$

$$\boldsymbol{u} = \nabla \times \boldsymbol{\Psi} \quad (19)$$

$$\Delta \boldsymbol{\Psi} = -\boldsymbol{\omega} \quad (20)$$

The MHD equations

Compressible MHD

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} &= -\nabla p - \frac{1}{4\pi} \boldsymbol{B} \times \nabla \times \boldsymbol{B} \\
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B})
\end{aligned}
$$

Solutions

Different wave modes are solutions to the MHD equations

- Alfvén modes (incompressible, aligned to the magnetic field)
- Fast magnetosonic (compressible)
- slow magnetosonic (compressible)

The MHD equations

Incompressible MHD

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= 0 \\ \rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} &= -\nabla p - \frac{1}{4\pi} \boldsymbol{B} \times \nabla \times \boldsymbol{B} \\ \frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \end{aligned}$$

Solutions

Different wave modes are solutions to the MHD equations

- Alfvén modes (incompressible, aligned to the magnetic field)

The MHD equations

Elsasser variables

$$
\begin{aligned}
\boldsymbol{w}^- &= \boldsymbol{v} + \boldsymbol{b} - v_A \boldsymbol{e}_z \\
\boldsymbol{w}^+ &= \boldsymbol{v} - \boldsymbol{b} + v_A \boldsymbol{e}_z \\
\Rightarrow \boldsymbol{v} &= \frac{\boldsymbol{w}^+ + \boldsymbol{w}^-}{2} \\
\Rightarrow \boldsymbol{b} &= \frac{-\boldsymbol{w}^+ + \boldsymbol{w}^- - 2 v_A \boldsymbol{e}_z}{2}
\end{aligned}
$$

Solutions

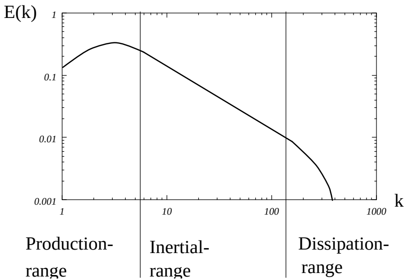
Different wave modes are solutions to the MHD equations

- Alfvén modes (incompressible, aligned to the magnetic field)
- $w^{\pm}$ correspond to forward/backward moving waves

Short review of Kolmogorov

- Kolmogorov assumes energy transport which is scale invariant
- Time-scale depends on eddy-turnover times
- Detailed analysis of the units reveals 5/3-law

Simplified picture – but enough to illustrate what we are doing next



Short review of Kolmogorov

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Simplified picture – but enough to illustrate what we are doing next

Dimensional Argument

5/3-law can be derived by assuming scale independent energy transport and dimensional arguments.
This gives a unique solution.

Kraichnan-Iroshnikov

- General idea: Turbulence is governed by Alfvén waves
- Collision of Alfvén waves is mechanism of choice
- Eddy-turnover time is then replaced by Alfvén time scale

Kraichnan-Iroshnikov

- Energy $E = \int (V^2 + B^2) dx$ is conserved
- Cascading to smaller energies
- Dimensional arguments cannot be used to derive $\tau_\lambda = \lambda / v_\lambda$
- Dimensionless factor v_λ / v_A can enter

Wave-packet interaction

$$\begin{aligned} \boldsymbol{w}^- &= \boldsymbol{v} + \boldsymbol{b} - v_A \boldsymbol{e}_z \\ \boldsymbol{w}^+ &= \boldsymbol{v} - \boldsymbol{b} + v_A \boldsymbol{e}_z \end{aligned}$$

Solutions: $w^\pm = f(\boldsymbol{r} \mp v_A t)$

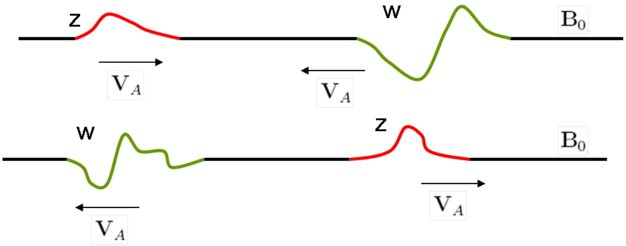
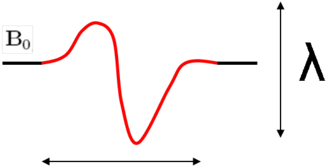


Figure: Packets interact, energy is conserved, shape is not

Kraichnan-Iroshnikov



Packet interaction

Amplitudes $\delta w_\lambda^+ \propto \delta w_\lambda^- \propto \delta v_\lambda \propto \delta b_\lambda$ during one collision

$$\Delta\delta v_\lambda \propto (\delta v_\lambda^2/\lambda)(\lambda/v_A)$$

Number of collision required to change wave packet

$$N \propto (\delta v_\lambda/\Delta\delta v_\lambda)^2 \propto (v_A/\delta v_\lambda)^2$$

$$\tau_{IK} \propto N\lambda/v_A \propto \lambda/\delta v_\lambda(v_A/\delta v_\lambda)$$

$$E_{IK} = \delta v_k^2 k^2 \propto k^{-3/2}$$

Goldreich-Sridhar

- 3/2 spectrum not that often observe
- Let's get a new theory
- We come back to Kraichnan-Iroshnikov: Alfvén waves govern the spectrum
- Turbulent energy is transported through multi-wave interaction
- Goldreich and Sridhar made up a number of articles dealing with this problem
- They distinguish between strong and weak turbulence

Goldreich-Sridhar

Weak Turbulence

What is weak turbulence?

Weak turbulence describes systems, where waves propagate if one neglects non-linearities. If one includes non-linearities wave amplitude will change slowly over many wave periods.
The non-linearity is derived pertubatevily from the interaction of several waves

Goldreich-Sridhar

Weak Turbulence

- Goldreich-Sridhar claim that Kraichnan-Iroshnikov can be seen as three-wave interaction of Alfvén waves
- They also believe, this is a forbidden process, since resonance condition reads

$$\begin{aligned} k_{z1} + k_{z2} &= k_z \\ |\, k_{z1} \,| + |\, k_{z2} \,| &= |\, k_z \,| \end{aligned}$$

- 4-wave coupling is now their choice!

Goldreich-Sridhar

Weak Turbulence

$$
\begin{aligned}
\boldsymbol{k}_1 + \boldsymbol{k}_2 &= \boldsymbol{k}_3 + \boldsymbol{k}_4 \\
\omega_1 + \omega_2 &= \omega_3 + \omega_4 \\
\Rightarrow k_{1z} + k_{2z} &= k_{3z} + k_{4z} \\
k_{1z} - k_{2z} &= k_{3z} - k_{4z}
\end{aligned}
$$

Parallel component unaltered $\Rightarrow$ 4-wave interaction just changes $k_\perp$ of quasi-particles

Weak Turbulence

$$\mid \delta v_\lambda \mid \sim \left| \frac{d^2 v_\lambda}{dt^2} (k_z v_A)^{-2} \right|$$

$$\frac{d^2 v_\lambda}{dt^2} \sim \frac{d}{dt}(k_\perp v_\lambda^2) \sim k_\perp v_\lambda \frac{dv_\lambda}{dt} \sim k_\perp^2 v_\lambda^3$$

since $(\boldsymbol{v} \cdot \nabla) \sim v_\lambda k_\perp$ for Alfvén waves

$$\left| \frac{\delta v_\lambda}{v_\lambda} \right| \sim \left(\frac{k_\perp v_\perp}{k_z v_\lambda} \right)^2 \Rightarrow N \sim \left(\frac{k_\perp v_\perp}{k_z v_\lambda} \right)^4$$

$$\sum v_\lambda^2 = \int E(k_z, k_\perp) d^3 k$$

$$\overset{\text{constant rate of cascade}}{\Rightarrow} E(k_z, k_\perp) \sim \epsilon(k_z) v_a k_\perp^{-10/3}$$

This spectrum now hold, when the velocity change is small

Goldreich-Sridhar

Strong Turbulence

Assume perturbation v_λ on scales $\lambda_\parallel \sim k_z^{-1}$ and $\lambda_\perp \sim k_\perp^{-1}$

$$\zeta_\lambda \sim \frac{k_\perp v_\lambda}{k_z v_\lambda} \quad \text{Anisotropy parameter}$$

$$N \sim \zeta_\lambda^{-4} \overset{\text{isotropic excitation} k_\perp \sim k_z \sim L^{-1}}{\sim} \left(\frac{v_A}{v_L}\right)^4 (k_\perp L)^{-4/3}$$

Everything is fine as long as $\zeta_\lambda \ll 1$

GS assume *frequency renormalization* when this is not given

Goldreich-Sridhar

Strong Turbulence

When energy is injected isotropically $\zeta_L \sim 1$ we have a critical balance

$\Rightarrow k_z v_A \sim k_\perp v_\lambda$

$\Rightarrow$ Alfvén timescale and cascading time scale match

For the critical balance and scale-independent cascade we find

$$k_z \sim k_\perp^{2/3} L^{-1/3}$$

$$v_\perp \sim v_A (k_\perp L)^{-1/3}$$

$$\Rightarrow E(k_\perp, k_z) \sim \frac{v_A^2}{k_\perp^{10/3} L^{1/3}} f\left(\frac{k_z L^{1/3}}{k_\perp^{2/3}}\right)$$

Goldreich-Sridhar

Strong Turbulence

Consequences

- A cutoff scale exists
- Eddies are elongated

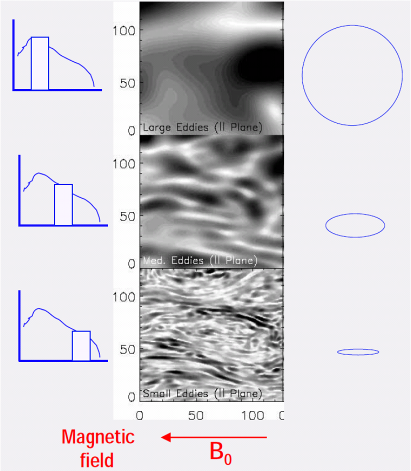


Figure: From Cho, Lazarian, Vishniac (2003)

Turbulence Conclusion

- There is a number of turbulence models out there
- It is not yet clear which is correct
- All analytical turbulence models are incompressible
- The only thing we know for sure is that there is power law in the spectral energy distribution

Numerical methods

Methods for the Navier-Stokes-equation

Projection	solve compressible equations, project on incompressible part
Vorticity	use vector potential to always fulfill divergence-free condition
Spectral	use Fourier space to do the same as above

Methods are presented for the Navier-Stokes-equation but also apply to MHD

Spectral methods will be presented for Elsässer variables in detail

Projection method

Different possible schemes
Here a scheme proposed by [1].

Step 1: Intermediate Solution

We are looking for a solution $\boldsymbol{u}^*$, which is not yet divergence free

$$\frac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} = -\left((\boldsymbol{u}\nabla)\boldsymbol{u}\right)^{n+1/2} - \nabla p^{n-1/2} + \nu\Delta\boldsymbol{u}^n \qquad (21)$$

This requires some standard scheme to solve the Navier-Stokes-equation (cf. Pekka's talk)

Projection method

Step 2: Dissipation

Use Crank-Nicholson for stability in the dissipation step

$$\nu \Delta \boldsymbol{u}^n \rightarrow \frac{1}{2} \nu \Delta \left(\boldsymbol{u}^n + \boldsymbol{u}^* \right) \tag{21}$$

yields

$$\boldsymbol{u}^{**} = \boldsymbol{u}^n - \Delta t \left((\boldsymbol{u}\nabla)\boldsymbol{u} + \nabla p \right) \tag{22}$$

$$\left(1 - \frac{\nu \Delta t}{2} \Delta \right) \boldsymbol{u}^* = \left(1 + \frac{\nu \Delta t}{2} \Delta \right) \boldsymbol{u}^{**} \tag{23}$$

Projection method

Step 3: Projection step

To project $\boldsymbol{u}^*$ on the divergence-free part, an auxiliary field ϕ is calculated, which fulfills

$$\nabla\cdot(\boldsymbol{u}^* - \Delta t \nabla\phi) = 0 \quad (21)$$

$$\Delta\phi = \frac{\nabla\cdot\boldsymbol{u}^*}{\Delta t} \quad (22)$$

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t \nabla\phi^{n+1} \quad (23)$$

Here a *Poisson* equation has to be solved. This is the tricky part and is the most time consuming.

Projection method

Step 4: Pressure gradient

Finally the pressure is updated. This can be calculated using the divergence of the Navier-Stokes-equation

$$p^{n+1/2} = p^{n-1/2} + \phi^{n+1} - \frac{\nu \Delta t}{2} \Delta \phi^{n+1} \tag{21}$$

Projection method

Total scheme

$$\frac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} + \nabla p^{n+1/2} = -[(u \cdot \nabla)\boldsymbol{u}]^{n+1/2} + \frac{\nu}{2}\nabla^2(\boldsymbol{u}^n + \boldsymbol{u}^*) \quad (21)$$

$$\Delta t \nabla^2 \phi^{n+1} = \nabla \cdot \boldsymbol{u}^* \quad (22)$$

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t \nabla \phi^{n+1} \quad (23)$$

$$\nabla p^{n+1/2} = \nabla p^{n-1/2} + \nabla \phi^{n+1} - \frac{\nu \Delta t}{2} \nabla \nabla^2 \phi^{n+1} \quad (24)$$

Vorticity-Streamline

The natural formulation for divergence-free flows is the stream function $\boldsymbol{u} = \nabla \times \boldsymbol{\Psi}$

Step 1: Calculate velocity

$$\boldsymbol{u}^n = \nabla \times \Psi^n \tag{25}$$

Vorticity-Streamline

Step 2: Solve the Navier-Stokes-equation

$$\partial_t \boldsymbol{u} = -\left(\boldsymbol{u}\nabla\right)\boldsymbol{u} + \nu\Delta\boldsymbol{u} \qquad (25)$$

Vorticity-Streamline

Step 3: Calculate Vorticity

$$\partial_t \omega = \nabla \times (\partial_t \boldsymbol{u}) \quad (25)$$

$$\omega^{n+1} = \omega^n + dt \cdot \Delta\omega \quad (26)$$

Vorticity-Streamline

Step 4: Calculate Streamfunction

Solve Poisson equation

$$\Delta\Psi^{n+1} = -\omega^{n+1} \quad (25)$$

Problem: This would result in a mesh-drift instability. Use staggered grid!

Vorticity-Streamline

Complete scheme

$$\mathbf{u} = \nabla \times \Psi \tag{25}$$

$$(u_x)_{i,j+1/2} = \frac{\Psi_{i,j+1} - \Psi i, j}{dy} \tag{26}$$

$$(u_y)_{i+1/2,j} = \frac{\Psi_{i+1,j} - \Psi i, j}{dx} \tag{27}$$

$$\mathbf{u}_{i+1/2,j+1/2} = \begin{pmatrix} 1/2\left((u_x)_{i+1,j+1/2} + (u_x)_{i,j+1/2}\right) \\ 1/2\left((u_y)_{i+1/2,j+1} + (u_y)_{i+1/2,j}\right) \end{pmatrix} \tag{28}$$

Spectral methods

Basic idea: Use Fourier transform to transform PDE into ODE
Problem: This is complicated for the nonlinear terms.

Spectral Navier-Stokes-equation

$$\frac{\partial \tilde{u}_\alpha}{\partial t} = -ik_\gamma \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)\left(\widetilde{u_\beta u_\gamma}\right) - \nu k^2 \tilde{u}_\alpha \tag{29}$$

$$k_\alpha \tilde{u}_\alpha = 0$$

Quantities with a tilde are fouriertransformed.
What's so special about the red and green term?

Spectral methods

The green term

Start with the Navier-Stokes-equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\nabla)\boldsymbol{u} + \nabla p = 0 \tag{29}$$

we can identify

$$ik_\gamma \delta_{\alpha\beta} \widetilde{u_\beta u_\gamma} = (\boldsymbol{u}\nabla)\boldsymbol{u} \tag{30}$$

and taking the divergence of the Navier-Stokes-equation

$$\Delta p = \nabla \cdot (\boldsymbol{u}\nabla)\boldsymbol{u} \tag{31}$$

so the second term is the gradient of the pressure

$$ik_\gamma \left(\frac{k_\alpha k_\beta}{k^2}\right) \left(\widetilde{u_\beta u_\gamma}\right) = -\nabla p \tag{32}$$

Spectral methods

Step 1: Fouriertransform

Starting with $\tilde{u}_\alpha$ the velocity has to be transformed into u_α
Then $u_\alpha u_\beta$ can be calculated (only 6 tensor elements are needed)
This is transformed back to Fourier space $\widetilde{u_\alpha u_\beta}$

Spectral methods

Step 2: Anti-Aliasing

In the high wavenumber regime of u_α we will find flawed data.
Due to the fact, that not all data is correctly attributed for by the Fourier transform, aliased data has to be removed
For $|\boldsymbol{k}| > 1/2k_{max}$ set $\widetilde{u_\alpha u_\beta}(\boldsymbol{k}) = 0$

Spectral methods

Step 3: Update velocity

$$\tilde{u}^*_\alpha = \tilde{u}^n_\alpha - \Delta t(ik_\gamma \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) (\widetilde{u_\beta u_\gamma}) - \nu k^2 \tilde{u}_\alpha) \tag{29}$$

Spectral methods

Step 4: Projection

Project u^*_α on its divergence free part

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \frac{\boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{u}^*)}{k^2} \tag{29}$$

Spectral methods

Why to use Spectral Methods?

Pro

- Easy to implement
- Relies on fast FFT-algorithms
- Easy to parallelize

Contra

- high aliasing-loss

Fourier Transforms

Fourier transforms are available in large numbers on the market

- **FFTW 2** ubiquitious library, allows parallel usage
- **FFTW 3** the improved version, *no parallel support*
- **Intel MKL** includes DFT, much faster than FFTW, simple parallelized version, limited to Intel-like systems
- **P3DFFT** UC San Diego's Fortran based FFT, highly parallelizable, requires FFTW-3
- **Sandia FFT** Sandia Lab's C based FFT, highly parallelizable, requires FFTW-2

Parallel Computing

Usually a parallel fluid simulation requires exchange of field borders between processors.
Spectral methods are slightly different:

- The calculation of $u_\alpha u_\beta$ and the update step are completely local. They have no derivatives and do not need neighboring points
- Every non-local interaction is done in the FFT

So the only thing we have to do, is using local coordinates and a parallel FFT

Extension to MHD

Elsässer notation

$$(\partial_t - v_A k_z)\, w_\alpha^- = \frac{i}{2}\frac{k_\alpha k_\beta k_\gamma}{k^2}\left(w_\beta^+ w_\gamma^- + w_\beta^- w_\gamma^+\right) - i k_\beta w_\alpha^- w_\beta^+ - \frac{\nu}{2} k^{2n} w_\alpha^-$$

$$(\partial_t + v_A k_z)\, w_\alpha^+ = \frac{i}{2}\frac{k_\alpha k_\beta k_\gamma}{k^2}\left(w_\beta^+ w_\gamma^- + w_\beta^- w_\gamma^+\right) - i k_\beta w_\alpha^+ w_\beta^- - \frac{\nu}{2} k^{2n} w_\alpha^+$$

The equations are treated similarly to the Navier-Stokes-equation .
Two major changes:

- Two coupled equations
- Magnetic field introduced via $-v_A k_z$ term

[1] D. L. Brown, R. Cortez, and M. L. Minion. Accurate Projection Methods for the Incompressible Navier-Stokes Equations. *Journal of Computational Physics*, 168:464–499, Apr. 2001.