Test-particle acceleration simulations

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Introduction

Energetic particles form a population in almost all collisionless plasmas from the magnetosphere to the interstellar medium. The usual models for their generation involve particle acceleration in electric fields by the Lorentz force through the usual equation of motion

$$\dot{\boldsymbol{p}} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}),$$

where $\boldsymbol{p} = \gamma m \boldsymbol{v}$, $\gamma = \sqrt{1 + p^2 m^{-2} c^{-2}}$ and $\boldsymbol{v} = \dot{\boldsymbol{r}}$ are the momentum, the Lorentz factor and the velocity of the particle, m and q are its mass and charge, and $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ are the electric and magnetic field in the system.

Since the electromagnetic fields, in general, are non-trivial functions of position and time in astrophysical plasmas, the solution of the equation of motion requires numerical methods. In this exercise, we will get acquainted with one of such methods, Monte Carlo simulations. We will limit the discussion to test-particle methods, which means that the electromagnetic fields are treated as prescribed in our treatment. Thus, particles can be treated fully independently from each other in the simulation.

The brute-force numerical method for modeling test-particle acceleration in astrophysical plasmas would start from a description of the electromagnetic fields and solve the equation of motion for a large number of particles in the system. However, in typical astrophysical scenarios, the primary length scale of charge-particle motion, i.e., the Larmor radius

$$r_L = \frac{p}{qB} = \frac{v}{\Omega}$$

is several orders of magnitude lower than the macroscopic length scale L of the fields. Here,

$$\Omega = \frac{qB}{\gamma m}$$

is the gyro frequency of the particle in the magnetic field. The maximal time step in a numerical solution of the full equation of motion would be a fraction of the gyro time $\tau = 2\pi/\Omega$. Clearly, τ would be orders of magnitude smaller than the dynamical time scale L/v. For these reasons, it would be computationally unfeasible to construct a global model of the astrophysical object tracing the particle motion through the system in a complete manner.

A way to circumvent the problem of inefficient time stepping is to introduce the guiding center approximation. Then, one is not interested in the actual trajectory of the particle but rather on the trajectory of its guiding center, $\boldsymbol{r}_{GC}(t)$, which is obtained by averaging the particle orbit over the rapid gyromotion in the magnetic field. This approximation is valid if the lowest scale lengths of the electromagnetic fields are much larger than the Larmor radius of the particles.

The guiding center motion can be broken into components parallel and perpendicular to the local magnetic field. In the non-relativistic case, the equation of motion parallel to the magnetic field reads

$$\dot{p}_{\parallel} = -\mathfrak{M}\nabla_{\parallel}B + qE_{\parallel},$$

where $\mathfrak{M} = \frac{1}{2}mv_{\perp}^2/B$ is the magnetic moment of the charged particle and

$$\nabla_{\parallel} = (\boldsymbol{B}/B) \cdot \nabla$$

is the spatial derivative in the field direction. In the perpendicular direction, the guiding center is drifting at velocity

$$\boldsymbol{v}_D = \frac{\boldsymbol{F} \times \boldsymbol{B}}{qB^2},$$

where $\boldsymbol{F} = q\boldsymbol{E} - \mathfrak{M}\nabla B - (mv_{\parallel}^2/B)\nabla_{\parallel}\boldsymbol{B}$ is the total force acting on the guiding center, averaged over a gyroperiod, in the (non-inertial) frame co-moving with the guiding center. In slowly varying electromagnetic field, the magnetic moment is an adiabatic invariant, i.e., an approximate constant of motion, which makes it possible to construct efficient numerical simulation codes capable of tracing particle motion through a large-scale astrophysical system.[1] (Now, the limit on the time step in the trajectory solver comes from the lowest scale length of the field and not from the Larmor radius.)

To arrive at the guiding center approximation, we made the assumption that the scale length of the fields is much larger than the Larmor radius of the particle. Apart from some localized regions (which can be treated separately in a simulation), this assumption is typically valid in an astrophysical plasma, apart from one component of the electromagnetic field: the turbulence. According to observations, turbulent magnetic fluctuations $\delta\boldsymbol{B}$ at all scales from global to kinetic permeate the space plasmas in practically all environments. Thus, it would seem invalid to resort to guiding center theory. However, it can be shown rather easily that if the amplitude of the magnetic fluctuations at a given time scale is clearly lower than the mean magnetic field (averaged over the time scale of the fluctuation), a perturbation approach called ***the quasilinear approximation*** is applicable. Then, the guiding center reaction to the fluctuations turns out to be a resonant one, so that only fluctuations fulfilling certain resonance conditions contribute to the motion of the particle. The random turbulent fluctuations of the electromagnetic field, thus, cause scattering of the charged particle, which can be modeled with an additional, stochastic term in the particle equation motion. Solving such stochastic differential equations of guiding center motion is then the final task of the modeler of particle acceleration in an astrophysical plasma.

[1] Note: in a relativistic case, i.e., $p \gtrsim mc$, breaking the equation of motion in two parts as above is not as simple, but it is still straightforward to formulate the processes affecting the guiding center motion and to implement them as a numerical solver of guiding center motion.

Stochastic motion induced by turbulent fluctuations

As a first example, consider a simple case where there is no electric field and the magnetic field consists of a homogeneous background field and a fluctuating field causing isotropic scattering. How do you model the situation numerically?

First, consider the guiding-center motion in the homogeneous average field, $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z$. Thus,

$$\nabla_{\parallel} B = 0, \quad E_{\parallel} = 0$$

and the regular part of the equation of motion is simply

$$\dot{p}_{\parallel} = 0; \quad p = \text{const.}$$

It is customary to write $p_{\parallel} := p\mu$, where $\mu = \cos\alpha$ is the cosine of the particle's ***pitch angle***, i.e., the angle between the magnetic field and the velocity vector. Thus,

$$\dot{\mu} = 0; \quad v = \text{const.}; \quad \dot{z} = v\mu$$

and the particles move along straight lines in tz-plane.

Introducing the effect of the scattering to the equation of motion is the next step. Magnetic field fluctuations cause the particle pitch-angle to change in small random increments, which keeping the particle speed constant. (Exercise: think, why!) Thus, as a result of scattering, the particle trajectories in velocity space are random walk on a surface of a sphere, $|\boldsymbol{v}| = v = \text{const}$. It is well-known that random walk leads to diffusion.

A three-dimensional diffusion equation in velocity space, in general, reads

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\mathsf{D} \cdot \frac{\partial n}{\partial \boldsymbol{v}} \right),$$

where D is the diffusion tensor and $n(\boldsymbol{v}, t) = dN/d\boldsymbol{v}$ is the density of particles in the 3-D velocity space. If the diffusion would be isotropic, $\mathsf{D} = D\mathsf{I}$ with D constant, we could write

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial \boldsymbol{v}^2} = \frac{D}{v^2} \left[\frac{\partial}{\partial v} \left(v^2 \frac{\partial n}{\partial v} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial n}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 n}{\partial \varphi^2} \right],$$

where we have used spherical coordinates (v, θ, φ). In our case, particles stay on spherical surfaces $|\boldsymbol{v}| = v = \text{const.}$, so there's no diffusion in the "radial" direction v. Thus, for constant D on each spherical surface,

$$\frac{\partial n}{\partial t} = \nu \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial n}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 n}{\partial \varphi^2} \right],$$

where $\nu = D/v^2$ is the angular diffusion coefficient on the spherical surface under investigation. Note that the choice of polar direction was arbitrary. Thus, we may choose it to be the direction of the particle velocity vector at $t = 0$ and, thus, solve the equation with the initial condition that all N particles are at $\theta = 0$ at $t = 0$. In this case, n does not depend on φ, so it suffices to solve

$$\frac{\partial n}{\partial t} = \frac{\nu}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial n}{\partial\theta}\right) \approx \frac{\nu}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial n}{\partial\theta}\right).$$

Here, the approximation applies at small θ, i.e., at early times, when the diffusion has not yet spread the particle population appreciably. This equation can be solved as

$$n\,\sin\theta\,d\theta\,d\varphi \approx \frac{N}{4\pi\nu t}\exp\left\{-\frac{\theta^2}{4\nu t}\right\}\theta\,d\theta\,d\varphi$$

meaning that θ^2 is exponentially and φ uniformly distributed.

The angular diffusion coefficient can be determined using quasilinear theory and given as a function of the amplitude of the magnetic fluctuations roughly as

$$\nu \sim \Omega\frac{\langle\delta B_L^2\rangle}{B_0^2},$$

where $\langle\delta B_L^2\rangle$ is the intensity of magnetic fluctuations at wavelengths close to the Larmor radius of the particles.

We now have a recipe to solve the stochastic motion of N particles in the fluctuating field from $t = 0$ to $t = T$:

1. Choose the time step $\Delta t = T/K$, where K is a large enough integer so that $4\nu\Delta t \ll 1$ and put the particle counter $i = 0$.
2. Inject a new particle at $z = z_0$, $v = v_0$ and $\mu = \mu_0$, and update the particle counter $i \leftarrow i + 1$.
3. Move the particle in the z-direction using $z \leftarrow z + v\mu\Delta t$
4. Scatter the particle:
    (a) pick random numbers $\theta^2 = -4\nu\Delta t\,\ln R_1$ and $\varphi = 2\pi R_2$, where $R_1 \in (0,1]$ and $R_2 \in (0,1]$ are random numbers picked from uniform distribution. θ and φ are the scattering angles relative to the present particle propagation direction.
    (b) Use spherical trigonometry to calculate the new pitch-cosine
    $$\mu \leftarrow \mu\cos\theta + \sqrt{1-\mu^2}\sin\theta\cos\varphi$$
5. Update the time $t \leftarrow t + \Delta t$
6. If $t < T$, go to 3
7. If $i < N$, go to 2
8. End

As can be noticed, the algorithm to solve the particle motion in a fluctuating magnetic field is extremely simple. It can be easily generalized to include an inhomogeneous large-scale magnetic field, a large-scale electric field or both. Including fluctuating electric fields is also possible, but a bit trickier (we will learn about how to include them during the course).

Pre-school exercises

To prepare for the school, prepare as many of the exercises below as possible.

1. Find out what is meant by quasilinear theory. Use, for example, lecture notes at `http://theory.physics.helsinki.fi/~xfiles/shockwave/07/` (Ch. 6); however, much simpler and less mathematical descriptions can be found as well. Google!

2. Draw a picture of the scattering geometry. Take the particle velocity before the scattering to be aligned with the polar direction (z-axis) and the particle velocity after the scattering to point to another direction. Take both vectors to have the same length. Identify the angles θ and φ in your plot. Now, draw the magnetic field vector in the same picture pointing at an arbitrary direction. Mark the pitch-angle of the particle before (α_1) and after (α_2) the scattering in the plot and try to derive the formula

$$\cos\alpha_2 = \cos\theta\cos\alpha_1 + \sin\theta\sin\alpha_1\cos\varphi$$

   between the pitch angles.

3. Add a homogeneous electric field in the direction of the homogeneous magnetic field in the example above and update the algorithm to account for that. Note that now v is no longer a constant. You may, however, assume that scattering off magnetic fluctuations occurs instantaneously so that during each scattering v stays constant.

4. Read or skim through ***Jones & Ellison*** (1991): *Plasma physics of shock acceleration.* Space Science Reviews **58**, 259–346.[2]

Group exercise topic

We will code a simple model of particle acceleration at shocks and investigate its properties.

[2]Freely available at ADS.