Propagation of the lightning signal over a non-planar inhomogeneous earth

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Abstract–A method for calculating the direction errors in lighting location systems is developed. The method is restricted to smooth earth profiles and the conductivity of the earth must be high. The latter condition is usually well satisfied. Some numerical results are presented, too. It is found that the direction errors caused by local irregularities such as ridges, hills and conductivity anomalies are small, less than 1 degree for earth profiles typical for Southern Finland.

1. INTRODUCTION

In this paper we develop a general approximate method for computing the electromagnetic field of a lightning stroke in case of a non-planar inhomogeneous earth. The emphasis is in the effect of the earth, so we use a simple dipole model for the source. The analytical computations are performed in frequency domain, but obtaining numerical results in time domain is also easy.

The application we have in mind is the location of a lightning flash by means of several direction finder stations and the study of possible errors due to anomalies of the earth. A typical direction finder has two vertical loop antennas, perpendicular to each other, that measure the x and y components of the magnetic field. When a direction finder observes a lightning stroke, the reported direction is perpendicular to the direction of the measured horizontal magnetic field. The actual location of the stroke can be determined if at least two direction finders report an observation. In principle, the method requires that the earth is planar and homogeneous. It is our aim now to study direction errors caused by a non-planar and inhomogeneous earth.

The essential quantities required to calculate the direction errors are the horizontal derivatives of the vertical component of the electric field, since the horizontal components of the electric field are negligible.

2. THEORY

2.1 APPROXIMATE BOUNDARY CONDITION FOR $\partial E_z/\partial z$

Consider a slightly non-planar earth with a high conductivity (i.e. consider frequencies such that $\sigma >> \omega\varepsilon$) that varies slightly in the horizontal direction. The symbols $\sigma,\varepsilon,\omega$ and μ_o denote the conductivity and permittivity of the earth, the angular frequency and the vacuum permeability, respectively. Let us use a rectangular coordinate system (x,y,z) (Fig. 1) with z-axis vertically upwards. Denote the surface of the earth by $z = \zeta(x,y)$. The coordinate z is chosen so that $\zeta(x,y) < 0$ for all x and y. Define the slopes of the earth profile by

$$\gamma_1 = \frac{\partial\zeta(x,y)}{\partial x}, \qquad \gamma_2 = \frac{\partial\zeta(x,y)}{\partial y}. \tag{2.1.1}$$

It is assumed that $|\gamma_1|, |\gamma_2| << 1$, as is the case in a non-mountainous terrain. We will perform all calculations to first order in γ_1 and γ_2.

If the frequency is high enough, we can assume a source-independent surface impedance when formulating the boundary conditions (Wait, 1980). The surface impedance η is defined by the relation

$$\mathbf{E}_t = \eta\, \mathbf{H}\times\hat{\mathbf{n}} \tag{2.1.2}$$

where $\mathbf{E}_t$ is the tangential electric field, $\mathbf{H}$ is the magnetic field strength and $\hat{\mathbf{n}}$ is a unit vector normal to the surface of the earth. The quantity η is small compared to its vacuum value $\eta_o = \mu_o c$, and we will perform the calculations to first order in η, too. The vector $\hat{\mathbf{n}}$ points from the air towards the earth. To first order, it is given by

$$\hat{\mathbf{n}} = (\gamma_1,\gamma_2,-1). \tag{2.1.3}$$

Utilizing the x and y components of the equation $\hat{\mathbf{n}}\times\mathbf{E} = \hat{\mathbf{n}}\times\mathbf{E}_t$, where $\hat{\mathbf{n}}\times\mathbf{E}_t$ is obtained from (2), we get

$$E_x = -\eta H_y - \gamma_1 E_z, \qquad E_y = \eta H_x - \gamma_2 E_z \tag{2.1.4}$$

(again, to first order in η and γ).

We are interested in the boundary conditions of $\partial E_z/\partial z$ on the surface z=0. Because this surface is entirely in the air, which is assumed to be a vacuum, Maxwell's equations assume the forms

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla\times\mathbf{E} = -i\omega\mu_o\mathbf{H}, \quad \nabla\times\mathbf{H} = i\omega\varepsilon_o\mathbf{E} \tag{2.1.5}$$

at z = 0. Using (4) and (5) we can calculate $\partial E_z/\partial z$ as follows:

$$
\begin{aligned}
\frac{\partial E_z}{\partial z} &= -\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} \\
&= \frac{\partial}{\partial x}(\gamma_1 E_z) + \frac{\partial}{\partial y}(\gamma_2 E_z) + \eta\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) + H_y\frac{\partial \eta}{\partial x} - H_x\frac{\partial \eta}{\partial y} \\
&= \frac{\partial}{\partial x}(\gamma_1 E_z) + \frac{\partial}{\partial y}(\gamma_2 E_z) + i\omega\varepsilon_o\eta E_z + \frac{\partial \eta}{\partial x}\left(\frac{i}{\mu_o\omega}\nabla\times\mathbf{E}\right)_y - \frac{\partial \eta}{\partial y}\left(\frac{i}{\mu_o\omega}\nabla\times\mathbf{E}\right)_x
\end{aligned} \tag{2.1.6}
$$

To first order we have

$$
\begin{aligned}
(\nabla\times\mathbf{E})_y &= \partial_z E_x - \partial_x E_z \approx -\partial_x E_z \\
(\nabla\times\mathbf{E})_x &= \partial_y E_z - \partial_z E_y \approx \partial_y E_z
\end{aligned} \tag{2.1.7}
$$

because for example the term $\partial_z E_x$ can be calculated from Eqs. (4) and (5) as

$$
\partial_z E_x = -\eta\partial_y H_z - i\omega\varepsilon_o\eta E_x - \gamma_1\partial_z E_z \tag{2.1.8}
$$

and all terms on the right-hand-side of Eq.(8) are second order. Substituting (7) to (6) we thus obtain the result

$$
\frac{\partial E_z}{\partial z} = \frac{\partial}{\partial x}(\gamma_1 E_z) + \frac{\partial}{\partial y}(\gamma_2 E_z) + i\omega\varepsilon_o\eta E_z - \frac{i}{\omega\mu_o}\left(\frac{\partial \eta}{\partial x}\frac{\partial E_z}{\partial x} - \frac{\partial \eta}{\partial y}\frac{\partial E_z}{\partial y}\right), \quad z=0 \tag{2.1.9}
$$

This result has the following very useful property: its right-hand-side contains only E_z and its horizontal derivatives. This equation is equivalent to Wait's result (Wait, 1964, appendix, Eq. 18) if we replace $\partial E_z/\partial x$ and $\partial E_z/\partial y$ by $-ik_{ox}E_z$ and $-ik_{oy}E_z$, respectively. This is permissible within our approximation but not necessary, so we leave Eq. (8) as it stands.

Within our approximation, we can calculate the surface impedance η from the formula

$$
\eta = i\,\eta_o\frac{k_o}{k} \tag{2.1.10}
$$

(Wait, 1964) where $k_o = \omega/c$ and k is the (possibly coordinate dependent) propagation constant of the earth:

$$
k = \sqrt{\omega^2\mu_o\varepsilon - i\omega\mu_o\sigma} \tag{2.1.11}
$$

Now the surface impedance is assumed to be independent of the source field; Eq. (9) can be derived e.g. by calculating the surface impedance according to Eq. (2) using a plane wave source model and homogeneous earth.

2.2 APPROXIMATE INTEGRAL EQUATION FOR E_z

To be able to use Eq. (2.1.9), we shall derive an integral equation for E_z on the plane z=0. We use Green's theorem, valid for all functions Φ and Ψ:

$$\int_V d\mathbf{x}\left(\Phi\nabla^2\Psi - \Psi\nabla^2\Phi\right) = \int_{\partial V} dS\left(\Phi\frac{\partial\Psi}{\partial n} - \Psi\frac{\partial\Phi}{\partial n}\right) \qquad (2.2.1)$$

The vector potential **A** in the Coulomb (or Lorentz) gauge satisfies the equation

$$-\left(\nabla^2 + k_o^2\right)\mathbf{A}(\mathbf{x}) = \mu_o\mathbf{j}_o(\mathbf{x}) \qquad (2.2.2)$$

where $\mathbf{x}\in V$ and V is entirely within a vacuum. Here, $\mathbf{j}_o$ is the primary current density, which is assumed to be a vertical dipole in this paper. We introduce Green's function

$$G(R) \equiv \frac{1}{4\pi}\frac{e^{-ik_oR}}{R} \qquad (2.2.3)$$

which satisfies

$$-\left(\nabla^2 + k_o^2\right)G(\mathbf{x}\text{-}\mathbf{x}') = \delta(\mathbf{x}\text{-}\mathbf{x}') \qquad (2.2.4)$$

Using Eqs. (1) - (4) it is easy to derive the following formula (recall $\mathbf{E} = -i\omega\mathbf{A}$)

$$\mathbf{E}(\mathbf{x}) = \mathbf{E}_o(\mathbf{x}) + \oint_{\partial V} dS'\left(G(\mathbf{x}\text{-}\mathbf{x}')\frac{\partial}{\partial n'}\mathbf{E}(\mathbf{x}') - \mathbf{E}(\mathbf{x}')\frac{\partial}{\partial n'}G(\mathbf{x}\text{-}\mathbf{x}')\right) \qquad (2.2.5)$$

which is valid if $\mathbf{x}\in V$ and if V is entirely within a vacuum. The field $\mathbf{E}_o(\mathbf{x})$ is the primary field, defined by

$$\mathbf{E}_o(\mathbf{x}) = -\, i\omega\mu_o\int_V d\mathbf{x}'\; G(\mathbf{x}-\mathbf{x}')\; \mathbf{j}_o(\mathbf{x}') \qquad (2.2.6)$$

Taking $V = \{(x,y,z)|\; z\geq 0\}$, evaluating (5) at the point $\mathbf{x} = (x,y,z)$ and taking the limit $z\to 0+$ we obtain (the notation f(x,y) means f(x,y,0) for any f)

$$E_z(x,y) = E_{z0}(x,y) - \int dx'dy'\; G(x\text{-}x',y\text{-}y')\left[\frac{\partial E_z}{\partial z'}(x',y',z')\right]_{z'=0} + R(x,y) \qquad (2.2.7)$$

where we have defined

$$R(x,y) \equiv \lim_{z\to 0+} \int dx'dy' \; E_z(x',y') \left[\frac{\partial}{\partial z'} \; G(x-x',y-y',z-z')\right]_{z'=0} \qquad (2.2.8)$$

The calculation of R(x,y) (Appendix) yields the result

$$R(x,y) = \frac{1}{2} E_z(x,y) \qquad (2.2.9)$$

whence we obtain

$$E_z(x,y) = 2E_{z0}(x,y) - 2\int dx'dy' \; G(x-x',y-y') \left[\frac{\partial E_z}{\partial z'}(x',y',z')\right]_{z'=0} . \qquad (2.2.10)$$

The first term in the right-hand-side represents the direct term plus an image term. The bracket expression is to be substituted from Eq. (2.1.9). The integral term vanishes for a perfectly conducting planar earth, because then all terms on the right-hand-side of Eq. (2.1.9) are zero. In principle, (10) is an integral equation for E_z but, in accordance with our earlier approximations (nearly planar earth, high conductivity), we will replace E_z by $2E_{z0}$ in the integral term, i.e. we will use the Born approximation. Thus we obtain a scheme for calculating values of $E_z(x,y)$ for any electric structure of the earth by doing a two dimensional numerical integration over x' and y' for each point (x,y). According to Wait (1964, appendix), the expansion can also be carried out in such a way that the unperturbed situation is a planar earth with a constant high (but finite) conductivity.

3. APPLICATIONS

3.1 DIRECTION ERRORS

The direction vector reported by a direction finder is (modulo 180^{o}) proportional to $(\partial_x E_z, \partial_y E_z)$ in our approximation (see Eqs. (2.1.7) and (2.1.8)). This is fine as long as E_z is real, but as soon as E_z has an imaginary part, we have two vectors $(\partial_x \mathrm{Re}\, E_z, \partial_y \mathrm{Re}\, E_z)$ and $(\partial_x \mathrm{Im}\, E_z, \partial_y \mathrm{Im}\, E_z)$ with possibly different directions. Explicitly: the physical direction at any instant is in case of a harmonic time dependence

$$\nabla\, \mathrm{Re}\left(E_z e^{i\omega t}\right) = \cos\omega t\; \nabla\, \mathrm{Re}\, E_z - \sin\omega t\; \nabla\, \mathrm{Im}\, E_z \qquad (3.1.1)$$

so that the measured direction oscillates between two, generally unequal, directions. If e.g. $|\nabla \mathrm{Re}\, E_z| \gg |\nabla \mathrm{Im}\, E_z|$, the oscillation is such that the Re-direction dominates almost all the time. In case of a non-harmonic time dependence, the measured direction fluctuates between these two directions during the pulse, depending in a complicated way on both the time dependence of the pulse and the relative magnitudes of the quantities $\nabla \mathrm{Re}\, E_z$ and $\nabla \mathrm{Im}\, E_z$.

Let us consider the situation in greater detail in case of a harmonic time dependence. Introduce the shorthand notations $\mathbf{r} = \nabla\, \mathrm{Re}(E_z e^{i\omega t})$, $\mathbf{a} = \nabla\, \mathrm{Re}\, E_z$, $\mathbf{b} = -\nabla\, \mathrm{Im}\, E_z$. Eq. (1) is then rewritten as

$$\mathbf{r}(t) = \mathbf{a} \cos \omega t + \mathbf{b} \sin \omega t \tag{3.1.2}$$

which is the equation of an ellipse in the xy-plane (Fig. 2) with the major and minor axes $r_\pm$ given by

$$r_\pm = \left\{ \frac{1}{2}\left(a^2 + b^2\right) \pm \frac{1}{2}\sqrt{\left(a^2 - b^2\right)^2 + 4\left(\mathbf{a}\cdot\mathbf{b}\right)^2} \right\}^{1/2} \tag{3.1.3}$$

The direction θ_o of the major axis (the angle between x-axis and the major axis) is given modulo $\pi/2$ by

$$\theta_o = \frac{1}{2} \arctan\left[\frac{a^2 \sin 2\theta_a + b^2 \sin 2\theta_b}{a^2 \cos 2\theta_a + b^2 \cos 2\theta_b}\right] \tag{3.1.4a}$$

where θ_a and θ_b are the direction angles of **a** and **b**, respectively. We need θ_o determined modulo π. The ambiguity introduced by Eq. (4a) is resolved be requiring that

$$a^2 \cos 2\left(\theta_o - \theta_a\right) + b^2 \cos 2\left(\theta_o - \theta_b\right) > 0. \tag{3.1.4b}$$

The relations (4a) and (4b) can be obtained by maximizing $r(t)^2$ with respect to t.

The probability density that the observed angle is θ at an arbitrary instant of time is

$$P(\theta) = \frac{1}{\pi} \frac{ab \left|\sin\left(\theta_a - \theta_b\right)\right|}{a^2 \sin^2(\theta - \theta_a) + b^2 \sin^2(\theta - \theta_b)} \quad . \tag{3.1.5}$$

$P(\theta)$ is by definition proportional to $dt/d\theta$, where the function $t(\theta)$ is defined by Eq. (2). It is normalized to unity on an interval of length π. $P(\theta)$ has a maximum at $\theta = \theta_o$, so we will interpret θ_o as the direction reported by the direction finder. If $|\theta_a - \theta_b| \ll 1$, as is the case if our approximations are valid at all, the peak $P(\theta)$ can be approximated by the Lorentzian shape function

$$P(\theta) \approx \frac{1}{\pi} \frac{ab \left|\theta_a - \theta_b\right|}{a^2\left(\theta - \theta_a\right)^2 + b^2\left(\theta - \theta_b\right)^2}\,, \qquad \theta \text{ close to } \theta_a, \theta_b. \tag{3.1.6}$$

In this approximation, the direction of the major axis becomes

$$\theta_o \approx \frac{a^2\theta_a + b^2\theta_b}{a^2 + b^2} \quad . \tag{3.1.7}$$

A useful measure for the peak width is the width $\Delta\theta$ at the half-maximum, defined by $P(\theta_o+\Delta\theta) = P(\theta_o)/2$:

$$\Delta\theta \approx \frac{ab\left|\theta_a - \theta_b\right|}{a^2 + b^2} \tag{3.1.8}$$

We call $\Delta\theta$ the *fluctuation angle*, since it is related to the practically intractable changes of the observed direction as a function of time (which is rotation around the fluctuation ellipse in case of a harmonic time dependence). The difference between the average observed angle θ_o and the direction angle θ_{lin} of a straight line connecting the point of observation to the point just below the source is called the (systematic) *error angle* $\theta_{err} = |\theta_o - \theta_{lin}|$ (cf. Fig. 3).

3.2 NUMERICAL RESULTS IN FREQUENCY DOMAIN

Three models are studied now. The frequency is always set equal to 100 kHz and the height of the source dipole to 300 m. The source current I_o is 100 kA.
Model 1 is a symmetric Gaussian hill of 100 m height. The earth profile is thus

$$\zeta(x,y) = (100\ \text{m}) \exp\left(-\frac{x^2 + y^2}{W^2}\right) - 100\ \text{m} \tag{3.2.1}$$

The constant 100 m is subtracted to meet the requirement $\zeta(x,y) \leq 0$. It does not affect the results, because only the derivatives of the earth profile appear in the equations. The earth is assumed to be an ideal conductor. The calculation is performed using the values 1 km and 2 km for the width W.
Model 2 is a Gaussian ridge with a profile function

$$\zeta(x,y) = H \exp\left(-\frac{x^2}{(30\ \text{km})^2} - \frac{y^2}{W^2}\right) - H \tag{3.2.2}$$

The height H is 50 m or 100 m and the width W is one of the values 0.7 km, 1 km, 1.5 km and 2 km.
Model 3 is a Gaussian conductivity and permittivity anomaly of the form

$$\sigma(x,y) = \sigma_o \exp\left(\frac{x^2 + y^2}{(3\ \text{km})^2}\right)$$

$$\varepsilon(x,y) = \varepsilon_o\left(1 + 80\ \exp\left(-\frac{x^2 + y^2}{(3\ \mathrm{km})^2}\right)\right) \tag{3.2.3}$$

with $\sigma_o = 0.05\ 1/(\Omega\ \mathrm{m})$.

The error angle and the fluctuation angle are calculated on a rectangular mesh in the xy-plane, with -30 km < x,y < 30 km. The number of mesh points is 400. The results are plotted as a three-dimensional surface plot, where the height of the surface corresponds to the absolute magnitude of the fluctuation or error angle. Unless explicitly mentioned, the stroke point is (x = -4 km, y = 3.6 km).

Table 1 summarizes the parameters used in computations. The results are presented in Figs. 4-7.

In model 1 (Figs. 4-5) we see that the systematic and fluctuation errors have roughly the same magnitude which is less than 0.1^o outside the anomaly (the peaks at the center of the figure). Fig. 4 suggests that the errors are largest in sectors that lie approximately perpendicular to a straight line between the stroke and the anomaly region. This is not quite true in Fig. 5 (Fig. 5 is the same as Fig. 4 except that the stroke point is farther and that the hill is larger), where it can be seen that these "sectors" tend at an angle of 90^o to each other. It is interesting to note that the interference pattern can change drastically if a couple of parameters are changed. In any case, the magnitude of the errors is probably too small to have any practical significance in model 1.

Going on to model 2 (Figs. 6-7), it is found that the magnitudes of the errors are somewhat larger than in model 1. Also, because of the different shape of the anomaly, the regions of higher errors are somewhat modified. The magnitudes of the errors are approaching experimental limits: a (systematic) error of 1^o causes the stroke to be mislocated by nearly 2 km if the stroke is 100 km apart. It would be, of course, interesting to study even higher ridges and find out when the effects become large. The assumptions behind our approximations, however, do not permit this. In particular, the Born approximation for the solution of the integral equation might no longer be valid. Dropping the Born approximation would lead to very serious numerical difficulties: inverting even a sparse $N^2 \times N^2$ matrix (to solve a two-dimensional integral equation directly) where N is about 100 (the number of spatial mesh points) is a formidable task.

Fig. 8 displays the attenuation of the field magnitude for run 5 (cf. Table 1). In order for our approximations to be valid, this quantity must be <<1 everywhere.

Finally, in Fig. 9 we see the results for a conductivity/permittivity anomaly (model 3). In this model, the errors are very small.

3.3 NUMERICAL RESULTS IN TIME DOMAIN

It is straightforward to obtain results in the time domain by performing the above calculations with many values of ω and using a discrete Fourier transformation. The calculations are about 256 time as laborious as in the frequency domain since we used a 256-point Fourier transform. Because of this, only one preliminary example of time domain calculations is presented in this paper. We use a simple Gaussian time dependence for the source:

$$f(t) = \exp\left(-\left(\frac{t}{\tau}\right)^2\right) \qquad (3.3.1)$$

with the risetime $\tau = 3.9\ \mu s$. Eq. (1) is plotted in Fig. 10, with the time axis shifted for convenience. This model does not describe well the actual time dependence of a lightning stroke, but we are mainly interested in the rising part, i.e. the range $-\tau < t < 0$ or so in Eq. (1), because it is known that the direction finders take the most information during this time. The usual double-exponential form for the pulse (e.g. Uman and McLain, 1969, LeVine and Meneghini, 1978) is not suitable because it yields to a dipole moment which does not vanish when $t \to \infty$.

Fig. 11 displays the horizontal components of the magnetic field as functions of time. The coordinates of the stroke and the observation point are (x = -32.5 km, y = 23 km) and (x = 86 km, y = -32 km), respectively. We use a Gaussian ridge of length 25 km, width 1 km and height 100 m (i.e. the earth profile function is given by Eq. (3.2.2) where H and W are replaced by 100 m and 1 km, respectively, and the value 30 km in the denominator is replaced by 25 km). Because of retardation effects, the time axis is not directly comparable to Fig. 10. Fig. 12 displays the fluctuation of the observed direction,which is assumed to be perpendicular to the momentary horizontal magnetic field, as a function of time. The time axis is comparable to Fig. 11.

It is seen from Fig. (12) that the fluctuation in the direction depends crucially on the absolute magnitude of the magnetic field: when the field is weak, the fluctuation is large (several degrees) and vice versa. This implies that the direction finder should extract its direction information at the high-field phases of the pulse to give accurate direction estimates. The low-field phases can still be used to reject pulses emitted by cloud-to-cloud discharges and other unwanted sources of pulses.

4. CONCLUSIONS

The theory presented in this paper shows that we can calculate numerically the direction errors for any earth profile and for any conductivity/permittivity distribution, as far as the profile is smooth and gently sloping (in the sense $\gamma_1, \gamma_2 << 1$) and the conductivity is large (in the sense $\sigma >> \omega\varepsilon$). Removing either or both of these assumptions would lead to much more difficult calculations, almost impossible in practice.

The numerical results show that hills and conductivity/permittivity anomalies produce only relatively small direction errors (usually much less than 1^{o}). The sharp peaks present in some of the figures are located at the anomalies themselves. Our calculation need not be reliable there, but the peaks indicate the importance of placing direction finders in locally homogeneous and flat areas, an experimentally well-known fact. The earth profiles used in the calculations are typical to Southern Finland. Studying more severe anomalies is not possible due to our approximations. Systematic and fluctuation errors are usually of the same magnitude; sometimes the fluctuation error is somewhat larger. The errors depend very much on the direction. Although the anomalies studied here sometimes change the field magnitudes by more than ten percent, the direction errors remain small, usually less than one degree.
Following is a list of parameters whose effect need to be studied:

- the location of the observer (well studied)
- the frequency (not studied, one value used)
- the height of the source dipole (not studied, one value used)
- the location of the source (studied in case of a ridge)
- the type of the anomaly: conductivity, permittivity, or earth profile (partly studied)
- the "strength" of the anomaly, e.g. the absolute height and the slopes of the hill (partly studied: slope is more important than height)
- numerical parameters: integration mesh (studied though not documented here)

It is clear that because of numerous parameters involved, the subject is still open for future research.

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APPENDIX: Derivation of result (2.2.9)

Consider the integral

$$R(x,y,z) \equiv \int dx'dy' \;\; E_z(x',y') \left[\frac{\partial}{\partial z'} \;\; G(x-x',y-y',z-z')\right]_{z'=0} \qquad (A.1)$$

Using (2.2.3), this can be rewritten as

$$R(x,y,z) = z\int dx'dy' \;\; E_z(x',y') \left[(1 + ik_o R)\frac{G(R)}{R^2}\right]_{R = \sqrt{(x-x')^2 + (y-y')^2 + z^2}} \qquad (A.2)$$

Make the following change of variables:

$$\begin{aligned} x' &= x + z\,u \cos\varphi \\ y' &= y + z\,u \sin\varphi \end{aligned} \qquad (A.3)$$

so that (A.2) becomes

$$R(x,y,z) = z^3 \int_0^\infty du\, u \int_0^{2\pi} d\varphi\, E_z(x+zu \cos\varphi, y+zu \sin\varphi)(1+ik_o z\sqrt{1+u^2})\frac{G(z\sqrt{1+u^2})}{z^2(1+u^2)} \qquad (A.4)$$

Passing to the limit $z \to 0+$ we obtain

$$R(x,y) = \frac{1}{2} E_z(x,y) \int_0^\infty \frac{du\, u}{(1+u^2)^{3/2}} = \frac{1}{2} E_z(x,y) \qquad (A.5)$$

as stated in the text.