Electrodynamics, spring 2003

Exercise 9 (Thu 3.4., Fri 4.4.)

1. An infinite uniform sheet current having a harmonic time dependence produces a plane wave. Show this starting from the solution of the wave equation in Lorenz gauge, when the current density is $\mathbf{J}(\mathbf{r},t) = Ke^{-i\omega t}\ \delta(z)\ \mathbf{e}_y$ (K = constant) and the charge density is zero. You may encounter infinite terms, but they are not a problem for a physicist.

2. a) Show that the plane wave field $\mathbf{E} = E_0\mathbf{e}_x \sin(kz - \omega t)$ satisfies the wave equation in vacuum.
   b) Derive a potential representation for the wave in Lorenz gauge.

3. The half-space $z > 0$ is the air and the half-space $z < 0$ is the earth, whose permeability is μ_0 and in which there are only Ohmic currents (constant conductivity σ). We have earlier learned that a temporally slowly varying magnetic field obeys the diffusion equation in the earth: $\nabla^2\mathbf{B} - \mu_0\sigma\partial\mathbf{B}/\partial t = 0$. Assume that the field at the earth's surface is time-harmonic: $\mathbf{B}(z = 0, t) = B_0 e^{-i\omega t}\,\mathbf{e}_x$ (B_0 constant). Calculate the magnetic and electric field in the earth.

4. The electric field $\mathbf{E}(\mathbf{r},t)$ and the electric displacement can be given as Fourier integrals:
$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\ \mathbf{E}(\mathbf{r},\omega) e^{-i\omega t}$$
   and $\mathbf{D}(\mathbf{r},t)$ in the same manner.
   a) If the permittivity only depends on frequency then $\mathbf{D}(\mathbf{r},\omega) = \epsilon(\omega)\mathbf{E}(\mathbf{r},\omega)$. What is then the relationship between $\mathbf{D}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$?
   b) In a simple version of the model by Drude and Lorentz
$$\epsilon(\omega) = \epsilon_0 (1 + \frac{ne^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma})$$
   where $\gamma > 0$. Prove that the relationship derived in a) is causal, i.e. that $\mathbf{D}$ at time t only depends on previous (and simultaneous) values of $\mathbf{E}$. You will need calculus of residues to solve this problem.

5. Assume that ψ satisfies Helmholtz scalar equation $\nabla^2\psi + (\omega/c)^2\psi = 0$. Show that $\mathbf{E} = \mathbf{r} \times \nabla\psi$ satisfies Helmholtz vector equation $\nabla^2\mathbf{E} + (\omega/c)^2\mathbf{E} = 0$.

Return answers until Tuesday 1.4. at 14 o'clock