Applications of electrodynamics, spring 2005

Exercise 4 (Thursday 17.2., return answers until 16:00 on Monday 14.2.)

1. Determine the polarization and the total cross section of a perfectly conducting sphere whose radius is much smaller than the wavelength of the incident electromagnetic field.

2. a) Show that the Maxwell Garnett mixing formula for metallic inclusions \((\epsilon_i \to \infty)\) is

\[
\epsilon_{eff} = \epsilon_h \frac{1 + 2f}{1 - f}
\]

b) The Maxwell Garnett formula is accurate at low volume fractions \(f\). One attempt to improve its validity at larger \(f\) is to consider the following iterative process:

i) Start with a uniform host material of volume \(V_0\) and permittivity \(\epsilon_h\).

ii) Add a small amount \(\Delta V \ll V_0\) of metallic spheres and calculate the effective permittivity of this mixture.

iii) Consider the medium of the previous step as a uniform host material and add again inclusions of an amount \(\Delta V\). Continue this procedure several times.

Show that after \(N\) iterations

\[
\epsilon_{eff} = \epsilon_h \prod_{n=1}^{N} \frac{1 + \frac{2\Delta V}{V_0 + n\Delta V}}{1 - \frac{\Delta V}{V_0 + n\Delta V}}
\]

Show that at the limit \(N \to \infty\)

\[
\epsilon_{eff} = \frac{\epsilon_h}{(1 - f)^3}
\]

There is experimental evidence that this result is fairly accurate at least for \(0 \leq f \leq 0.5\).
3. Above a uniform earth, there is a horizontal electric dipole whose current density is $\mathbf{J}(r,t) = I L \delta(x) \delta(y) \delta(z - h)e^{-i\omega t}\mathbf{e}_z$. The electromagnetic parameters of the air are $\mu_0, \epsilon_0, \sigma_0$ and in the earth $\mu_1, \epsilon_1, \sigma_1$. The earth’s surface is the $xy$-plane.

a) Show that the vector potential can be expressed as

$$A_{x0} = C \int_0^\infty db \frac{b}{K_0} (e^{-K_0 |z-h|} + R(b)e^{-K_0 (z+h)}) J_0(bp), \ z > 0$$

$$A_{z0} = C \frac{\partial}{\partial x} \int_0^\infty db \ S(b)e^{-K_0 (z+h)} J_0(bp), \ z > 0$$

$$A_{x1} = C \int_0^\infty db \ P(b)e^{K_1 z} J_0(bp), \ z < 0$$

$$A_{z1} = C \frac{\partial}{\partial x} \int_0^\infty db \ Q(b)e^{K_1} J_0(bp), \ z < 0$$

where $C$ is a constant to be determined later and

$$K = \sqrt{b^2 - k^2}, \ \rho = \sqrt{x^2 + y^2}$$

b) Express $\mathbf{B}$ and $\mathbf{E}$ using only the vector potential.

c) Apply boundary conditions to solve $P, Q, R, S$.

d) Construct a line current of amplitude $I$ of successive dipoles. Calculate $A_{x1}$ and $A_{z1}$ (set $\mu_1 = \mu_0$).

e) Calculate $\mathbf{B}_1$ for a line current. Determine the constant $C$ by studying a time-independent current.

Reminder: there is still time to consider the extra problem no. 5 of exercise 3.